Languages of Markov Chains

Based on:

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On Regularity of unary Probabilistic Automata STACS 2016

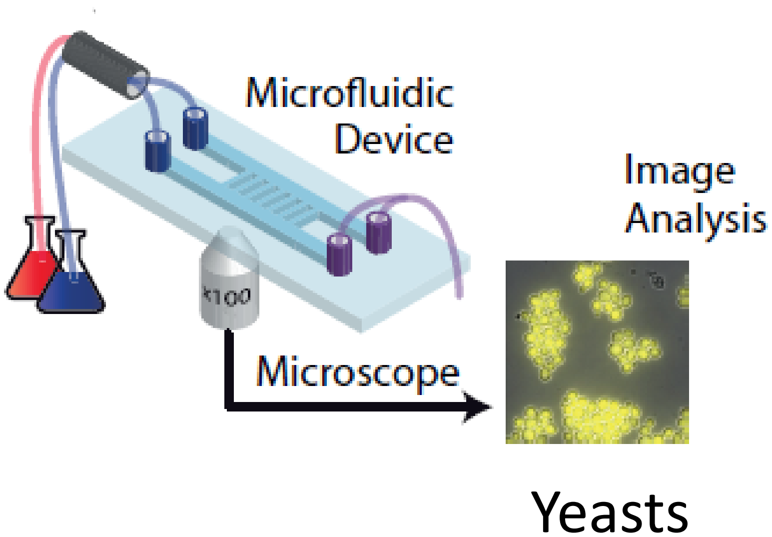
Manindra Agrawal, S. Akshay, Blaise Genest, P.S. Thiagarajan.

Approximate Verification of the Symbolic Dynamics of Markov Chains JACM'15-LICS'12

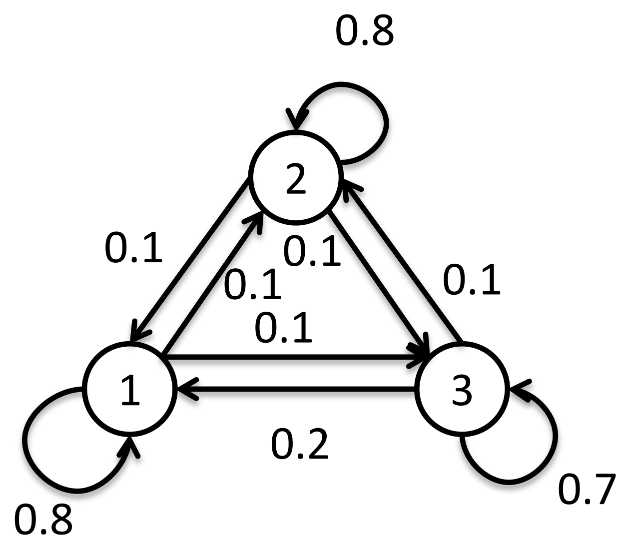
Overview of the Talk

- Biological Motivation
- Trajectories of Markov Chains
- Languages of Markov Chains
- Approximations

Population of cells



Stochastic
Model



Setting: Many yeasts. Simplistic model
Each yeast can be in one of 3 states (1 high, 2 med,3 low concentr. of X)

Experiments: Percentage of yeasts going from state S to state S’
=> Chance for a yeast to go from state S to state S’ (after 5 min).

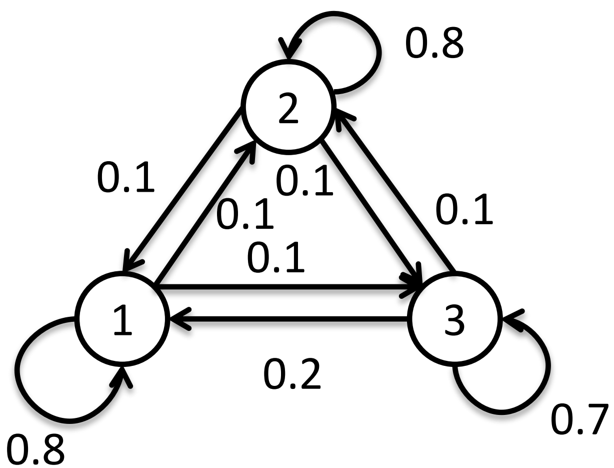
Image analysis: proportion in state 1 (high concentration of X => marker).

Continuous Abstraction

Non Continuous: Nathalie'sTalk tomorow.

Assume enough yeasts => Proportion in state 1,2,3.

Initial Proportion of cells : $P_{init} = \begin{pmatrix} - \\ \\ - \end{pmatrix}$ through image analysis



$$M_{yeast} = \begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}$$

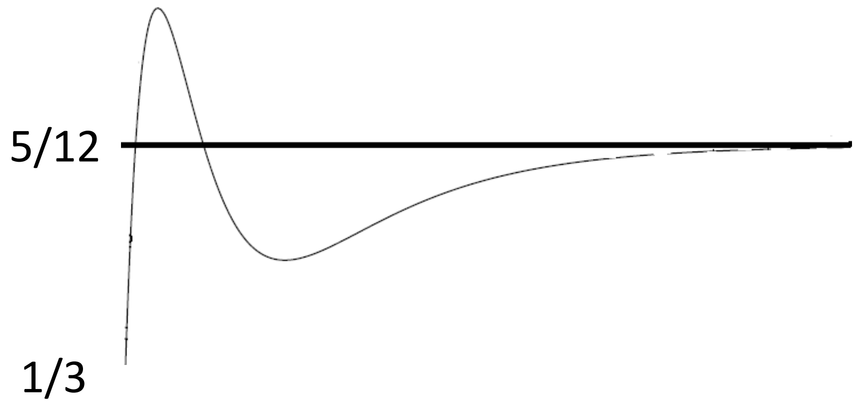
$P_{5min} = M_{yeast}\, P_{init}$ $P_{10min} = M^2_{yeast}\, P_{init}$ ....

Deterministic concrete trajectory from a given P_{init}

Symbolic Trajectory

Expermiments (image analysis):

first less than 5/12 of yeasts in state 1.
some time later more than 5/12 of yeasts in state 1.
then eventually, less than 5/12 of yeasts in state 1.



We set up Threshold=5/12

Below threshold: B
Above (or equal) threshold: A => finite alphabet {A,B}

So we observed $B^{n1}A^{n2}B^{\omega}$ Symb. Trajectory = infin. word on {A,B}

Language of Markov Chain: set of trajectories from Init={P_{init} | $x \in [0,2/3]$}

Quantitative Question

Is $B^{n1}A^{n2}B^{\omega}$ in the language of the Markov Chain for some n1,n2 ?

If yes, for which initial proportion, for which n1,n2?

i.e. for which subset of s in $\quad Init = \{ \begin{pmatrix} - \\ \\ - \end{pmatrix} \mid x \quad [\; -] \quad ?$

Looks like a verification Question.
Use algorithm for solving PCTL* questions on Markov Chains ?

Cannot be modeled with PCTL* [Beauquier Rabinovitch Slissenko CSL'02]

Skolem Problems

Actually, even with a unique initial configuration $P_{init} = \begin{pmatrix} & - & \\ & & \\ - & & \end{pmatrix}$

Trajectory of a Markov chain from P_{init} is B^{ω} ?

as hard as Skolem (question on linear rec. seq. (e.g. Fibonacci))
[Akshay, Antonopoulose,Ouaknine, Worrel, IPL'15]

Decidability? Open for > 40 years.
Decidable for <6 states
If dec. for 18 states, major breakthrough in diophantines approximations

Simple Markov Chains

Simple: Every Eigen value of Markov Chain has multiplicity 1.

-> Markov Chain is diagonalizable.

For simple markov chains,

Trajectory of a Markov chain from some P_{init} is B^{ω} ?

=> decidable for 10 states.

=> for more than 25 states, breakthrough in Diophantines approx

Decidable if Trajectory of a simple Markov chain from some P_{init} is wB^{ω} for some finite word w (ultimate positivity)

[Ouaknine Worrell ICALP'14 (best paper) & ICALP'14]

Eigen Basis

Simple:Markov Chains are diagonalizable.

$$M_{yeast} = \begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{pmatrix}$$

Express P_{init} in the eigen vector basis

$$M \cdot \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} = \underline{1} \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} \; ; \; M \cdot \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} = \underline{0.7} \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} \; ; \; M \cdot \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix} = \underline{0.6} \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix}$$

$$P_{init} = \begin{pmatrix} 1/3 \\ 1/4 \\ 5/12 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} - \frac{2}{5} \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix}$$

$$M^n P_{init} = \begin{pmatrix} 5/12 \\ 1/3 \\ 1/4 \end{pmatrix} + \frac{1}{5}\, 0.7^n \begin{pmatrix} 5/12 \\ -5/12 \\ 0 \end{pmatrix} - \frac{2}{5}\, 0.6^n \begin{pmatrix} 5/12 \\ 0 \\ -5/12 \end{pmatrix}$$

$M^n P_{init}\ [1] \geq$ — iff $-0.7^n \geq -0.6^n$

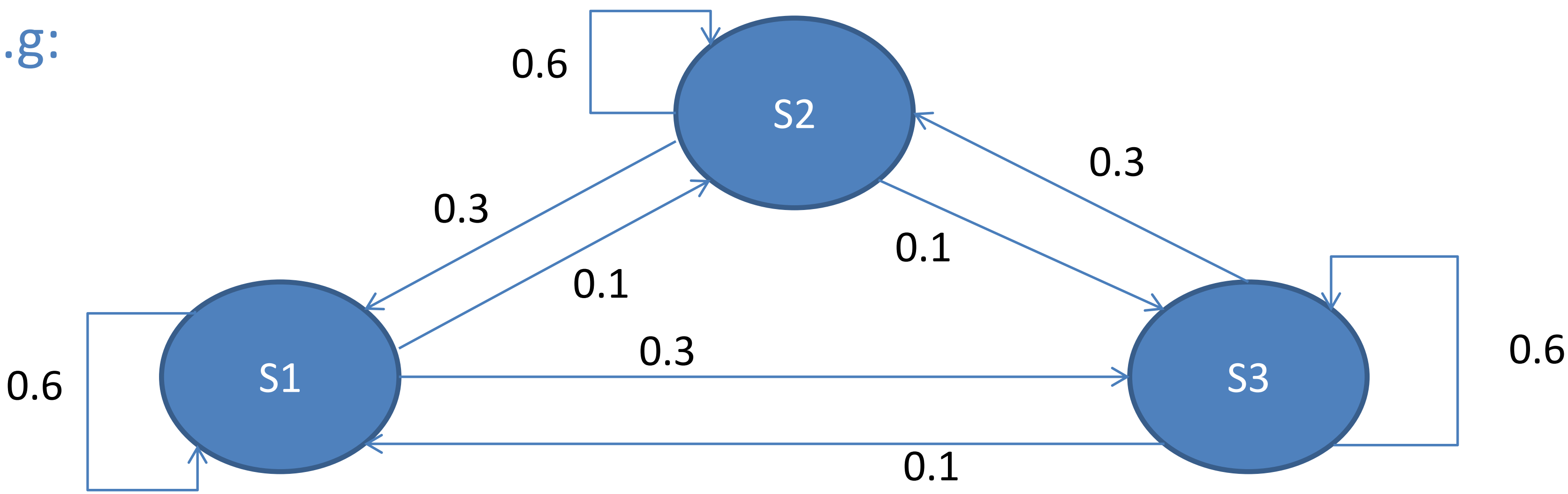
Symbolic Trajectory from P_{init}: $B^k A^\omega$

General Simple Markov Chains

$M^n P_{init}$ [1] $\geq \tau$ iff $a_0 + a_1 \sigma_1^n + \ldots + a_k \sigma_k^n \geq 0$

Trajectories are not always ultimately periodic, even for simple M

e.g:



$M^n P_{init}$ [0] $\geq 1/3$? with $P_{init}(S1) = P_{init}(S2) = 1/4$.

Reason: eigen values: 1, —— ——

Roots of real numbers

If eigen values are roots of real numbers,
then trajectories are ultimately periodic

$M^n P_{init}$ [1] $\geq \tau$ iff $a_0 + a_1 \sigma_1^n + \ldots + a_k \sigma_k^n \geq 0$

Let l_k with $\sigma_k^{l_k}$ is positive real,
Let $L = \mathrm{lcm}(l_k)$ and $\rho_k = \sigma_k^L$

$M^{Ln} P_{init}$ [1] $\geq \tau$ iff $a_0 + a_1 \rho_1^n + \ldots + a_k \rho_k^n \geq 0$

→ Eventually constant (dominant factor) e.g.: w A^ω

→ Ultimately periodic of period L
ex: L=5, trajectory w' $(A\ B\ B\ A\ B)^\omega$

Sum up

Property of eigenvalues of Markov chain	Ultimately periodic traj.
Distinct, positive real numbers	✓ even ult. constant
Distinct, roots of real numbers	✓ Decidability!
Distinct	× Decidability ?

but positivity/equality approximable

[Chadha Kini Viswanathan QEST’14]

What about languages?

set of trajectories from e.g. $\text{Init} = \{ \begin{pmatrix} - \\ \\ - \end{pmatrix} \mid x \quad [\quad -]$

Init needs to be a polytope.

Language: not that simple

M_{yeast}: eigen values are positive real numbers: 1, 0.7, 0.6

$$\begin{pmatrix}1/3\\1/4\\5/12\end{pmatrix}=\begin{pmatrix}5/12\\1/3\\1/4\end{pmatrix}+\frac{1}{5}\begin{pmatrix}5/12\\-5/12\\0\end{pmatrix}-\frac{2}{5}\begin{pmatrix}5/12\\0\\-5/12\end{pmatrix}$$

Trajectory: $B^k A^\omega$

$$\begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}=\begin{pmatrix}5/12\\1/3\\1/4\end{pmatrix}-\frac{1}{5}\begin{pmatrix}5/12\\0\\-5/12\end{pmatrix}$$

Trajectory: B^ω

When P_{init} converges towards $\begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}$

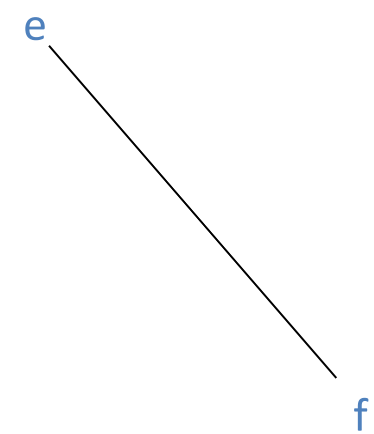
trajectory becomes $B^n A^\omega$ with n converging to ∞

=> Can show that language is $B^* A^\omega \cup B^\omega$

Not a finite union of trajectories

Language in general

Result: if all eigen values are distincts positive real numbers,
Then language is regular for Init a polytope.
[AGKV STACS'16]



e.g. : Set of Initial distributions:
$\{ \lambda e + (1-\lambda)f \mid \lambda \quad [0,1]\}$.

First, under these conditions, all trajectories are ultimately constant.

Ultimate Language

e A^ω

a_1(e)>0

$$\mathcal{L}_{ult}^{N_{max}}(H) = \{v \mid \exists w \in \{A,B\}^{N_{max}}, wv \in \mathcal{L}(H)\}$$

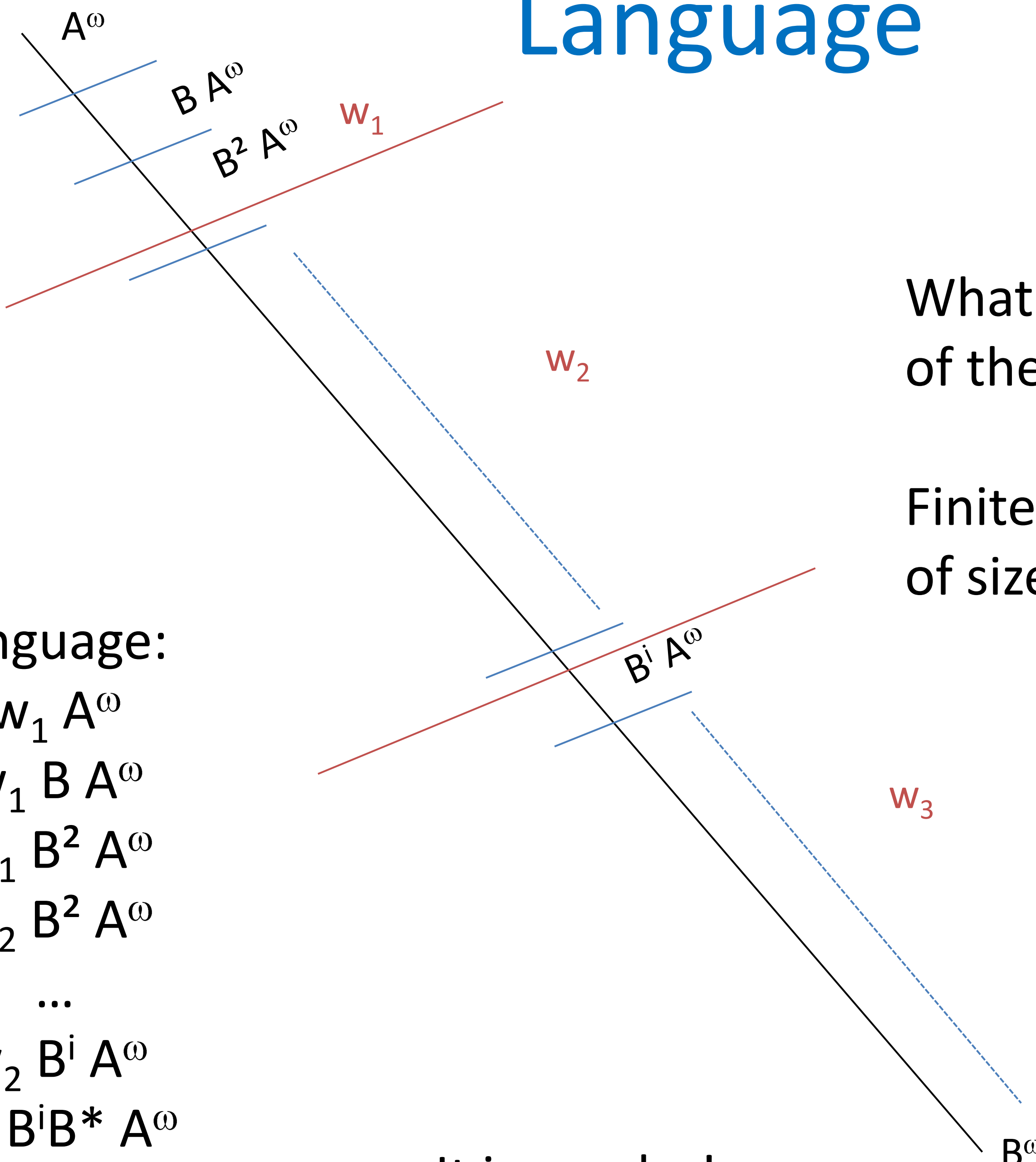
N_{max} such that after N_{max} steps,
the trajectories from e,f are A^ω and B^ω

The set of trajectories in (e,f) **after** N steps:
Lemma 1: Included into $B^* A^\omega$
Lemma 2: for all i, exists starting point with $B^i A^\omega$

a_1(f)=0
a_2(f)<0

f B^ω

Language



What about the prefixes of the N_{max} first steps?

Finite number of prefixes of size N_{max}.

Language:

$w_1\ A^{\omega}$

$w_1\ B\ A^{\omega}$

$w_1\ B^2\ A^{\omega}$

$w_2\ B^2\ A^{\omega}$

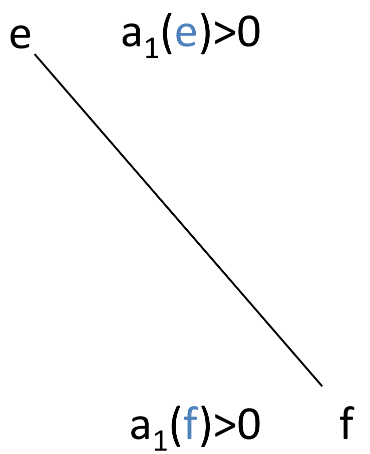
…

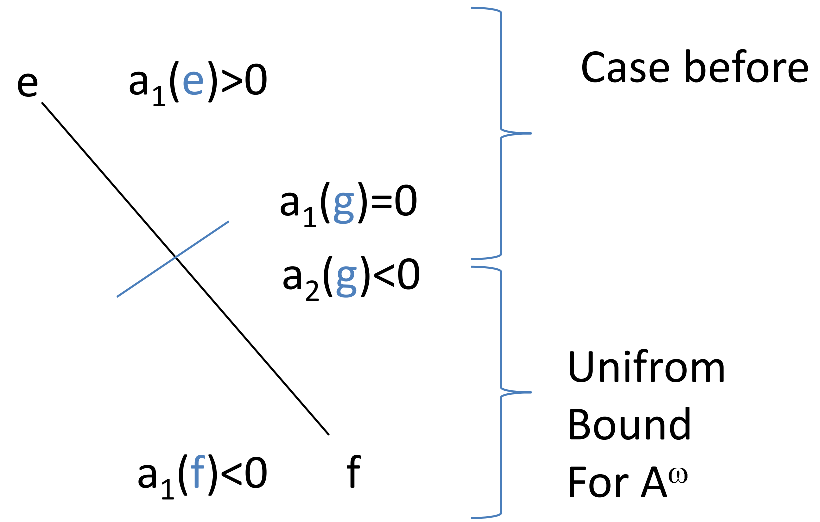
$w_2\ B^i\ A^{\omega}$

$w_3\ B^i B^*\ A^{\omega}$

It is regular!

In general with Polytope in 1D





max bound e, bound f
is a uniform bound
for utimately constant.

Polytopes in any Dimension

Case of $e_1..e_z$ extremities of Polytope with

$a_1(e_1) > 0$

$a_1(e_2)=0$, $a_2(e_2)<0$

$a_1(e_3)=a_2(e_3)=0$, $a_3(e_3)>0$

…

…. Sign $(a_k(e_k))=(-1)^k$

N is the max of the ultimately constant bound for $e_1..e_z$

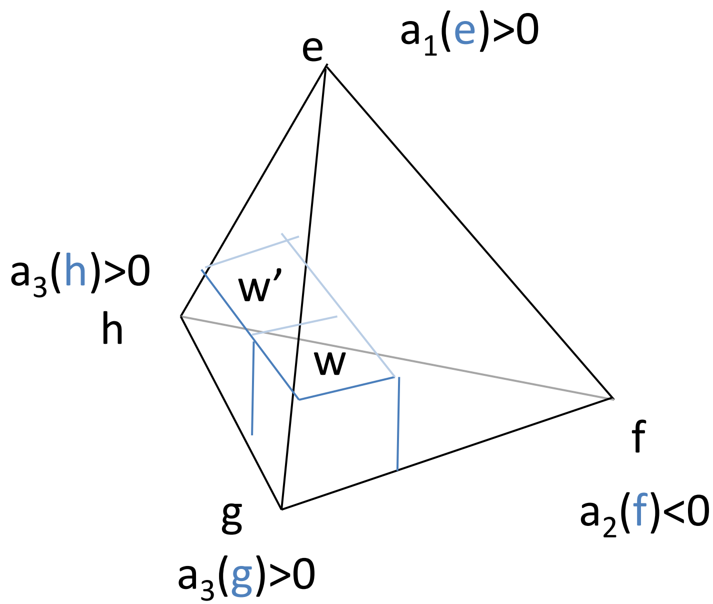
Result: $L^N_{ult}(H)= (A^*)\ B^*A^* \ldots B^*\ A^\omega$

The set of trajectories in $(e_1..e_z)$ **after** N steps:

Lemma 1: At most z switch, i.e. Included into $(A^*)\ B^*A^* \ldots B^*\ A^\omega$

Lemma 2: for all $i_1..i_z$, exists initial distrib with traj: $B^{i_1}\ A^{i_2} .. B^{i_z} A^\omega$

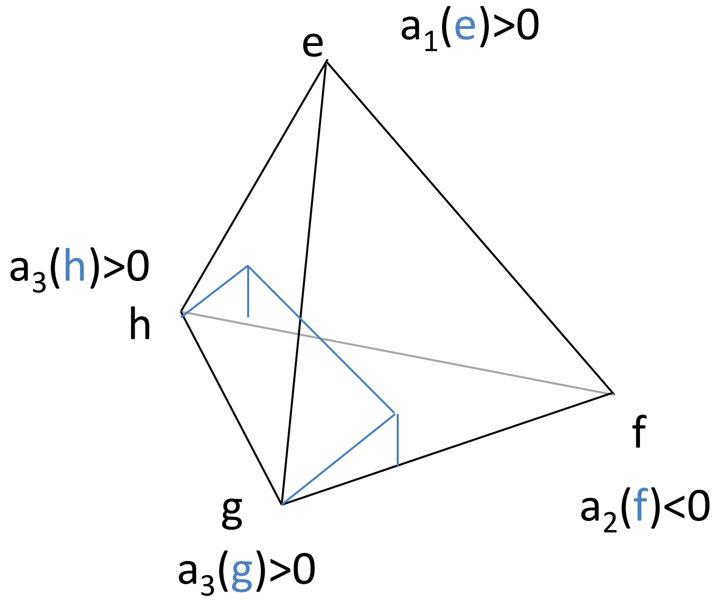
General Dimension: Handle Prefixes



Induction on the highest « z » in the space.

In the picture, z=3, n(dimension)=4
Take w touching (h,g) and touching (h,g,f) with a point not touching h or g
And touching (h,g,f,e) with a point not touching (hfg).
We can prove that for some i,
w A^i B A^ω is a trajectory continuity argument
w A^i B^ω is a trajectory => wA^iA*B*A^ω included into trajectory
w A^ω is a trajectory

In general



Induction on the highest « z » in the space.

Remove points with trajectory $wA^iA^*B^*A^\omega$ and $w'A^iA^*B^*A^\omega$
It remains a finite union of convex polyhedra with lower « z »

Hence the language is a finite union of regular set, hence it is regular.

Sum up

Property of eigenvalues of Markov chain	Regular language	Ultimately periodic traj.
Distinct, positive real numbers	✓ decidable	✓
Distinct, roots of real numbers	× Decidability ?	✓
Distinct	×	×

but approximable
[AAGT LICS'12]

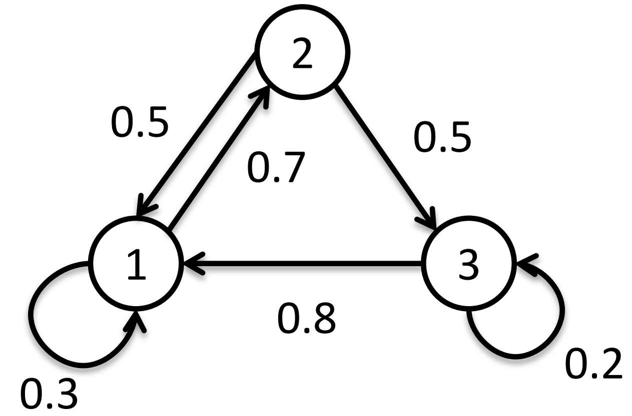
Distinct roots of real numbers: not regular.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{512} & \frac{8r+3}{512} & \frac{3+3r}{64} & \frac{13+16r}{128} & \frac{9+2r}{32} & \frac{1+4r}{16} & \frac{1-r}{2} \end{bmatrix}$$

Approximation for Markov Chains.

Irreducible aperiodic chains

$$M = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \\ 0.8 & 0 & 0.2 \end{pmatrix}$$



M is irreducible aperiodic because:

$$M^2 = \begin{pmatrix} .44 & .21 & .35 \\ .55 & .35 & 0.1 \\ .4 & .56 & .04 \end{pmatrix}$$

Approximations for irreducible aperiodic chains:

Irreducible aperiodic: unique stationary distribution **f**.

Fix ε => exists K such that $|M^K \mathbf{u} - \mathbf{f}| < \varepsilon$ for all initial distribution **u**.

$A_1..A_n$ is an (ε,K)-approximate symbolic trajectory of
a concrete trajectory $d_1..d_n$ if
$d_i \in A_i$ for all i<K and d_i is ε-close to A_i for i>K.

Exact symbolic trajectory from init: ABBABBBABBBBA…

Epsilon => K=4,

Approx symbolic trajectories:

ABBABBAA…, ABBABBAB…, ABBABBBA…, ABBABBBB….

We get ABBABB (A or B)* is regular.

Approximations for irreducible aperiodic chains:

Th: Given MC + Init (set), it is decidable [AAGT, LICS'12] whether:

For some concrete trajectory w, there does not exists a approx trajectory satisfying ϕ,

=> w does not satisfies ϕ.

=> system does not satisfy ϕ.

For all concrete trajectory w, all approx traj satisfy ϕ

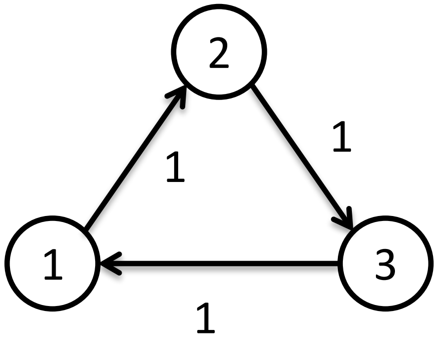
=> all w satisfies ϕ.

=> system satisfies ϕ.

Undetermined: for all concrete trajectory, there exists approximate trajectroty satisfying ϕ, but not for all.

=> Refine ε to reduce number of approx trajectories.

Irreducible Periodic chains

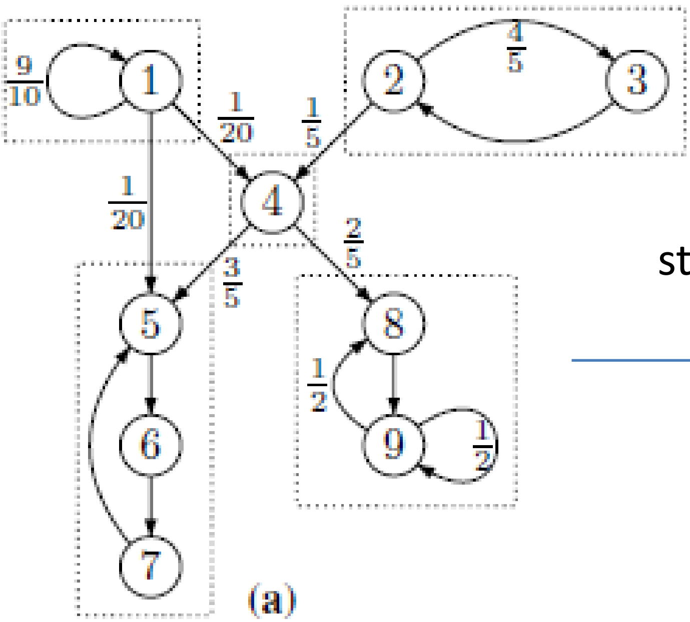


M is periodic of period 3.

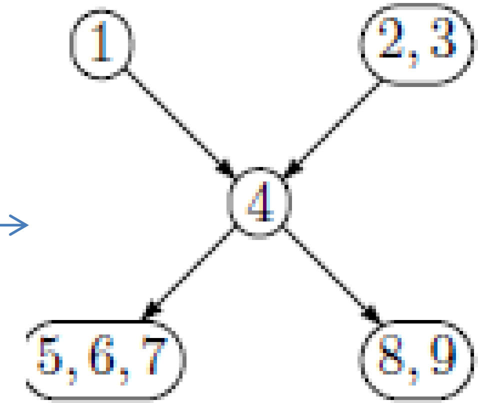
M^3 is irreducible aperiodic on disconnected partition of nodes.

Consider M^3 from Init,
Consider M^3 from M Init ,
Consider M^3 from M^2 Init

Not irreducible chains



Consider the strongly connected components.



Stationary distributions have weight 0 for non bottom SCC (1; 2-3, 4).
⇒Analyse the bottom SCC with earlier algorithm.

Tough part: Analyse non bottom SCC to get weights for bottom SCC, depending on Initial distribution (algorithm close to PCTL Mod. Check.)

+ uniform K over all initial distrib => allow to lift results to Languages

Polytope of initial Distributions

uniform K over all initial distrib => allow to lift results to Languages

Consider each extremities e1..en of the initial polytope.

Use linearity!

Compute the way they weight in the different BSCC

$e = \Sigma \lambda i\ ei$ => weight(e,BSCCk) = $\Sigma \lambda i$ weight(ei,BSCCk)

$\Rightarrow$Easy to compute the possible ultimately ε-reccuring set of letters

Then compute set of bounded prefixes with some ultimate set of letters, easy to compute as well.

Conclusion

Markov Chain (Unary PFA):
Simplistic formalism but still many open problems.

Even taking restrictive hypothesis,
not easy to describe their behavior.

But quantitative analysis of population is possible
under strong hypothesis or with approximations.