Constrained Sampling and Counting: When Practice Drives Theory

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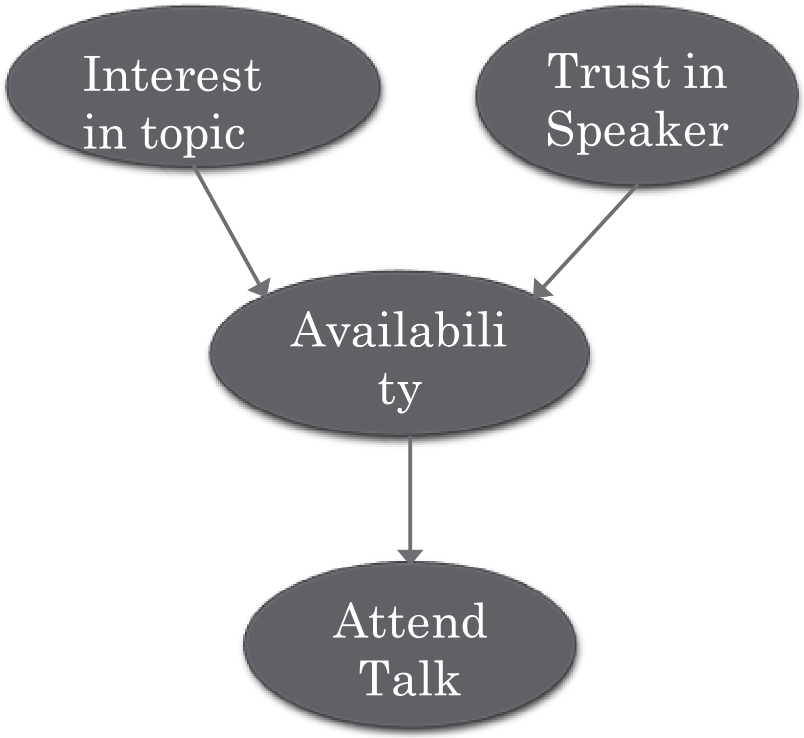
Probabilistic Inference

How do we infer useful information from the data filled with uncertainty?



Modeling Attendance for Today's Talk

\+

Pr(Attending Talk | Interest in topic = True)

Roth, 1996

Model Counting

Smart Cities

- Alarm system in every house that responds to either burglary or earthquake
- Every alarm system is connected to the central dispatcher (of course, automated!)
- Suppose one of the alarm goes off
- Important to predict whether its earthquake or burglary

Deriving Useful Inferences

What is the probability of earthquake (E) given that alarm sounded (A)?

$$\Pr[\text{event} \mid \text{evidence}]$$

Bayes' rule to the rescue

$$\Pr[E|A] = \frac{\Pr[E \cap A]}{\Pr[A]}$$

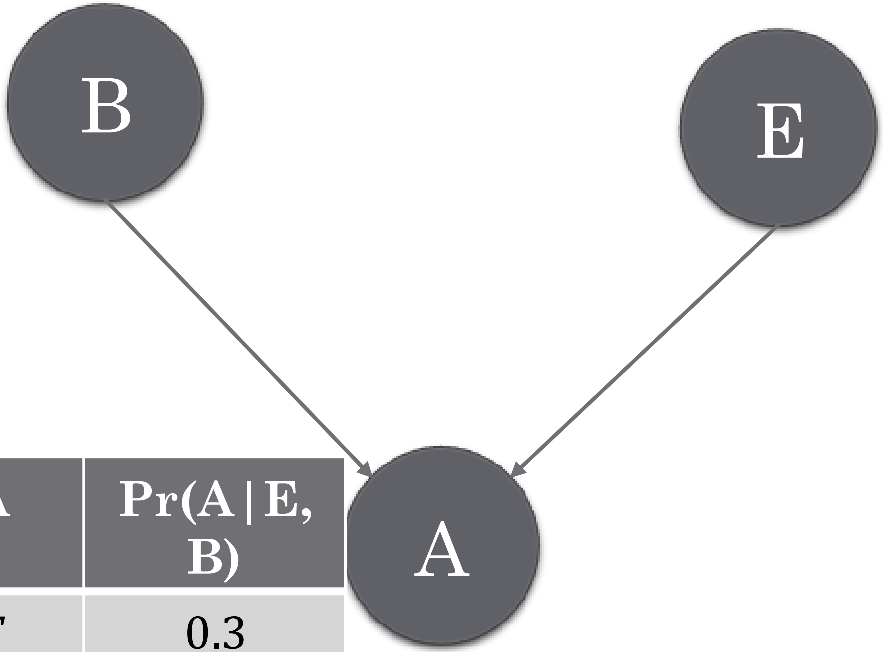
How do we calculate these probabilities?

Probabilistic Models

Graphical Models

Graphical Models

B	Pr
T	0.8
F	0.2

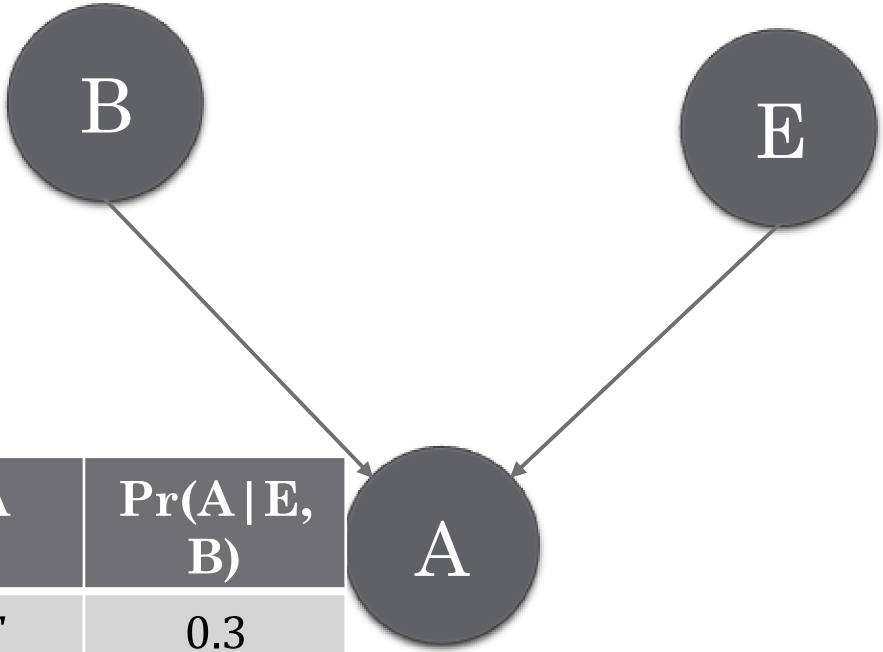


E	Pr
T	0.1
F	0.9

B	E	A	Pr(A|E, B)
T	T	T	0.3
T	T	F	0.7
T	F	T	0.4
T	F	F	0.6
F	T	T	0.2
F	F	F	0.8
F	F	T	0.1

Calculating $\Pr[E \cap A]$

B	Pr
T	0.8
F	0.2



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F	F	F	0.8
F	F	T	0.1

$$\Pr[E \cap A] \\ = \Pr[E] * \Pr[\neg B] * \Pr[A|E, \neg B] \\ + \Pr[E] * \Pr[B] * \Pr[A|E, B]$$

Calculating $\Pr[E \cap A]$

B	Pr
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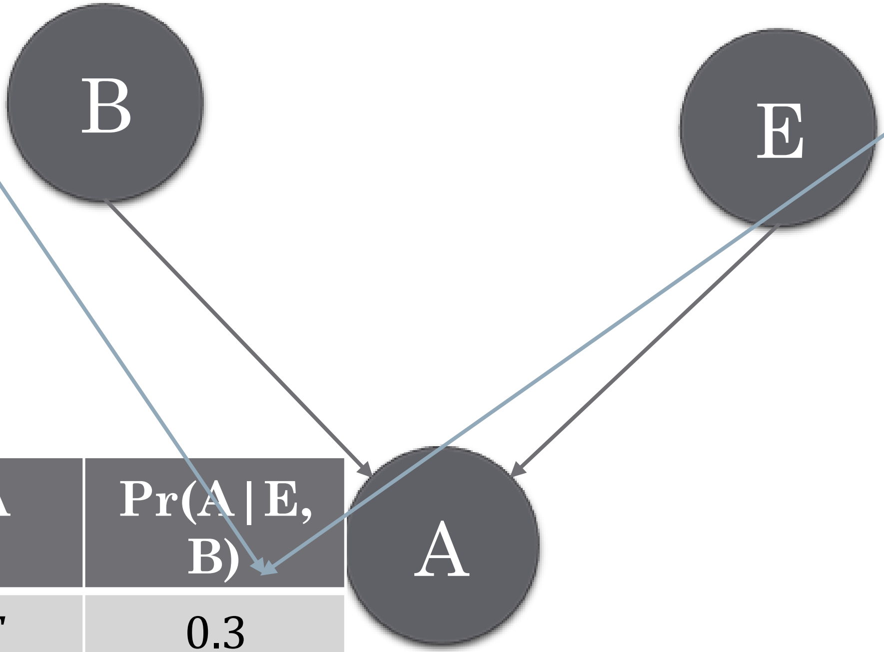

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$$\Pr[E \cap A]$$
$$= \Pr[E] * \Pr[B] * \Pr[A|E,B]$$
$$+ \Pr[E] * \Pr[\neg B] * \Pr[A|E,\neg B]$$

Calculating $\Pr[E \cap A]$

B	Pr
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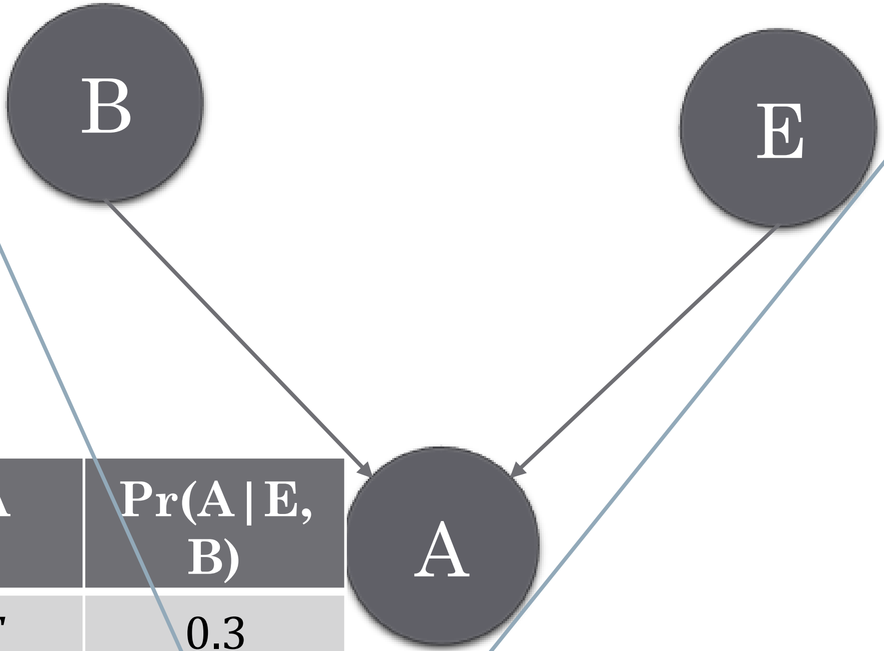

E	Pr
T	0.1
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B	**E**	**A**	**Pr(A|E, B)**
T	T	T	0.3
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T	F	T	0.4
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F	T	T	0.2
F	F	F	0.8
F	F	T	0.1

$$\begin{aligned}&\Pr[E \cap A] \\ &= \Pr[E] * \Pr[B] * \Pr[A|E,B] \\ &\quad + \Pr[E] * \Pr[\neg B] * \Pr[A|E,\neg B]\end{aligned}$$

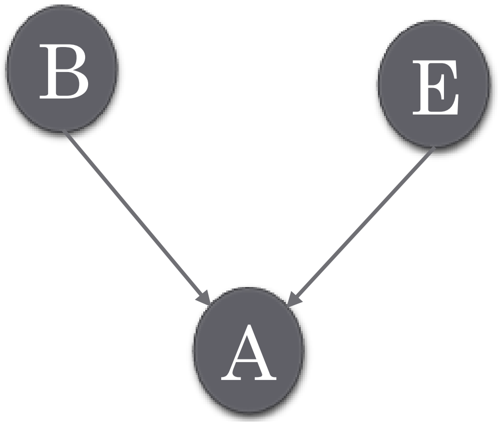
Moving from Probability to Logic

- $X = \{A, B, E\}$
- $F = E \wedge A$
- $W(B = 0) = 0.2, W(B = 1) = 1 - W(B = 0) = 0.8$
- $W(A = 0) = 0.1, W(A = 1) = 0.9$
- $W(E = 0 |\, A = 0, B = 0) = \cdots$
- $W(A = 1, E = 1, B = 1) = W(B = 1) * W(E = 1) * W(A = 1 | E = 1, B = 1)$
- $R_F = \{(A = 1, E = 1, B = 0), (A = 1, E = 1, B = 1)\}$
- $W(F) = W(A = 1, E = 1, B = 1) + W(A = 1, E = 1, B = 1)$

$$\boldsymbol{W(F) = \mathbf{Pr}[E \cap A]}$$

Weighted Model Count

Probabilistic Inference to WMC to Unweighted Model Counting



$\Pr[E|A]$ → Weighted Model Counting

Roth, 1996

Weighted Model Counting → Unweighted Model Counting

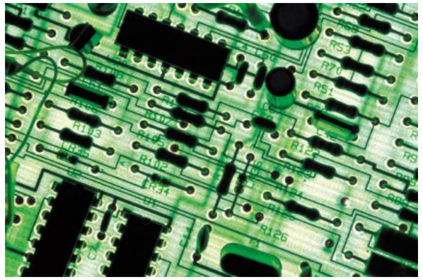
Polynomial time reductions

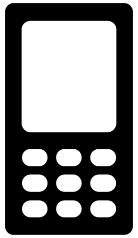
Model Counting

- Given a SAT formula F
- R_F: Set of all solutions of F
- Problem (#SAT): Estimate the number of solutions of F (#F) i.e., what is the cardinality of R_F?
- E.g., F = (a v b)
- $R_F = \{(0,1), (1,0), (1,1)\}$
- The number of solutions (#F) = 3

#P: The class of counting problems for decision problems in NP!

How do we guarantee that systems work *correctly* ?





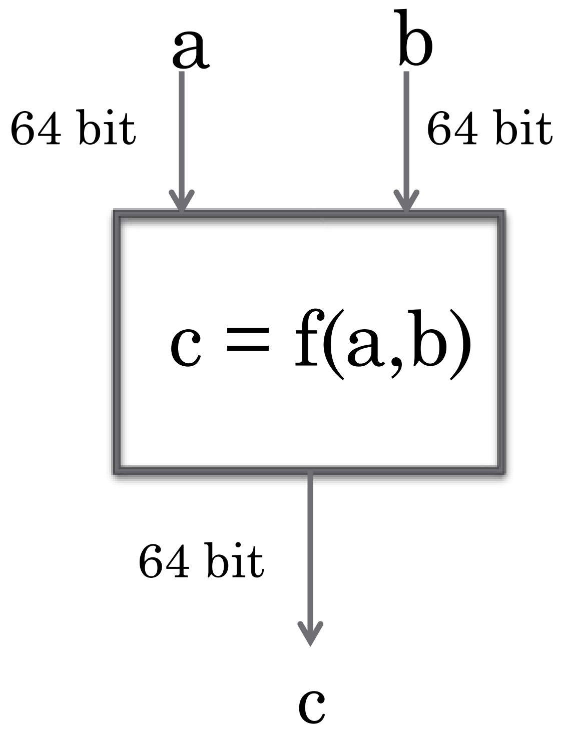
Functional Verification

- Formal verification
  - Challenges: formal requirements, scalability
  - ~10-15% of verification effort
- Dynamic verification: ***dominant approach***

Dynamic Verification

- Design is simulated with test vectors
  - Test vectors represent different verification scenarios
- Results from simulation compared to intended results
- **Challenge**: Exceedingly large test space!

Constrained-Random Simulation

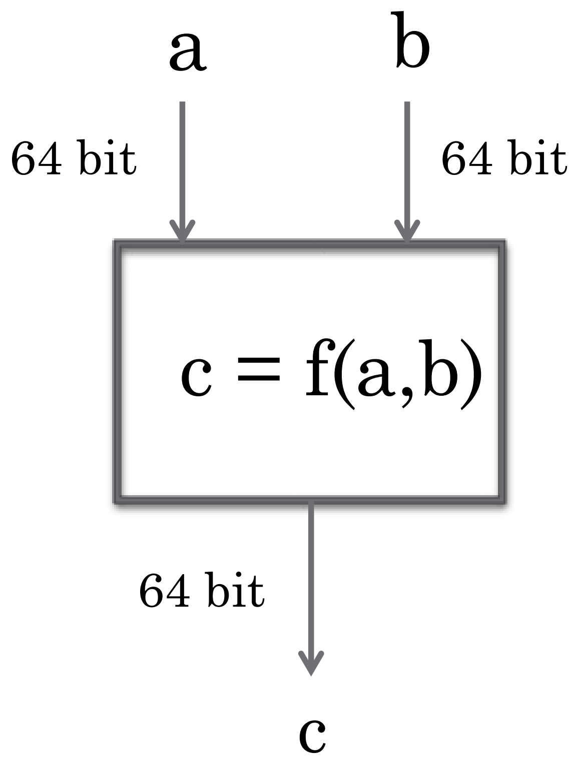


Sources for Constraints

- Designers:
  1. $a +_{64} 11 *_{32} b = 12$
  2. $a <_{64} (b >> 4)$
- Past Experience:
  1. $40 <_{64} 34 + a <_{64} 5050$
  2. $120 <_{64} b <_{64} 230$
- Users:
  1. $232 *_{32} a + b \ != 1100$
  2. $1020 <_{64} (b /_{64} 2) +_{64} a <_{64} 2200$

Problem: How can we *uniformly* sample the values of a and b satisfying the above constraints?

Problem Formulation



Set of
Constraints

↓

SAT Formula

↓

Sample satisfying assignments
uniformly at random

Agenda

Design Scalable Techniques for
Uniform Generation and
Model Counting
with Strong Theoretical Guarantees

Agenda

Design Scalable Techniques for
Almost-Uniform Generation and
Approximate-Model Counting
with Strong Theoretical Guarantees

Formal Definitions

- F: CNF Formula; $\mathrm{R_F}$: Solution Space of F
- Input: F Output: $y \in R_F$
- Uniform Generator:
  - Guarantee: $\forall y \in R_F$, $\Pr[y \text{ is output}] = \frac{1}{|R_F|}$
- Almost-Uniform Generator
  - Guarantee: $\forall y \in R_F$, $\frac{1}{(1+\varepsilon)|R_F|} \leq \Pr[y \text{ is output }] \leq \frac{(1+\varepsilon)}{|R_F|}$

Formal Definitions

- F: CNF Formula; $\mathrm{R_F}$: Solution Space of F

- Probably Approximately Correct (PAC) Counter
  - Input: F Output: C

$$\Pr\left[\frac{|R_F|}{(1+\varepsilon)} \leq C \leq |R_F|(1+\varepsilon)\right] \geq 1-\delta$$

Uniform Generation

Rich History of Theoretical Work

- Jerrum, Valiant and Vazirani (1986):
  - Uniform Generator: Polynomial time PTM (Probabilistic Turing Machine) given access to Σ_2^P oracle

Almost-Uniform Generator $\xleftrightarrow{\text{PTIME}}$ PAC Counter

Stockmeyer (1983): Deterministic approximate counting in 3rd level of polynomial hierarchy.
Can be used to design a BPP^NP procedure -- too large NP instances

No Practical Algorithms

Rich History of Theoretical Work

- Bellare, Goldreich, and Petrank (2000)
  - Uniform Generator: Polynomial time PTM given access to NP oracle
  - Employs n-universal hash functions

Universal Hashing

- $H(n, m, r)$: Set of r-universal hash functions from $\{0,1\}^n \to \{0,1\}^m$

  $\forall y_1, y_2, \cdots y_r$ (distinct) $\in \{0,1\}^n$ and $\forall \alpha_1, \alpha_2 \cdots \alpha_r \in \{0,1\}^m$

  $\Pr[h(y_i = \alpha_i)] = \frac{1}{2^m}$ (Uniformity)

  $\Pr[\, h(y_1 = \alpha_1) \wedge \cdots \wedge (h(y_r) = \alpha_r)] = 2^{-(mr)}$ (Independence)

- (r-1) degree polynomials → r-universal hash functions

Concentration Bounds

- t-wise ($t \geq 4$) random variables $X_1, X_2, \cdots X_n \in [0,1]$

$$X = \sum X_i \; ; \; \mu = E[X]$$

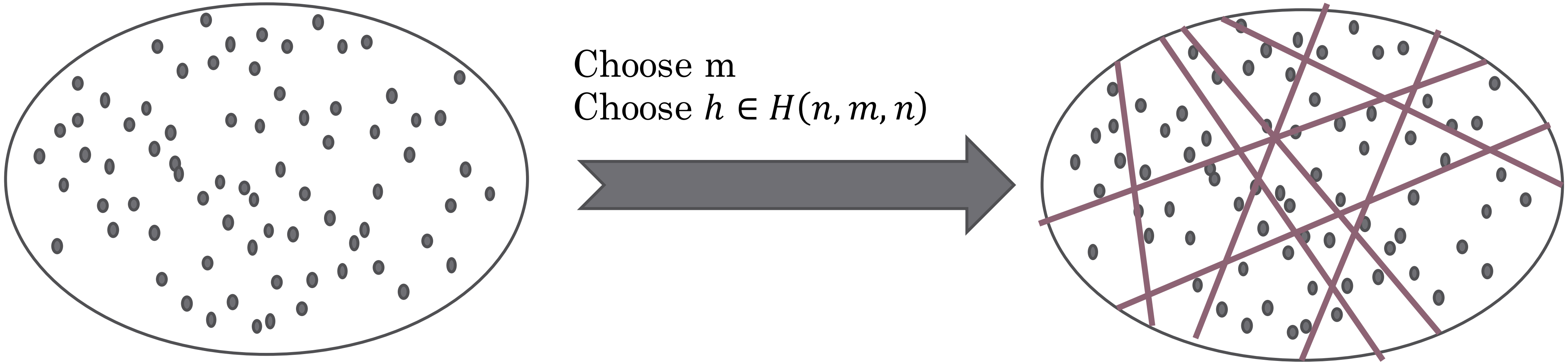
$$\Pr[\, |X - \mu| \leq A] \geq 1 - 8 \left(\frac{t\mu + t^2}{A^2}\right)^{\frac{t}{2}}$$

- For t = 2

$$\Pr[\, |X - \mu| \leq A] \geq 1 - \frac{\sigma^2[X]}{A^2}$$

BGP Method

- Polynomial of degree n-1
- SAT Solvers can not handle large polynomials!



- For right choice of m, all the cells are small (# of solutions $\leq 2n^2$)
- Check if all the cells are small (NP- Query)
- If yes, pick a solution randomly from randomly picked cell

In practice, the query is too long and can not be handled by SAT Solvers!

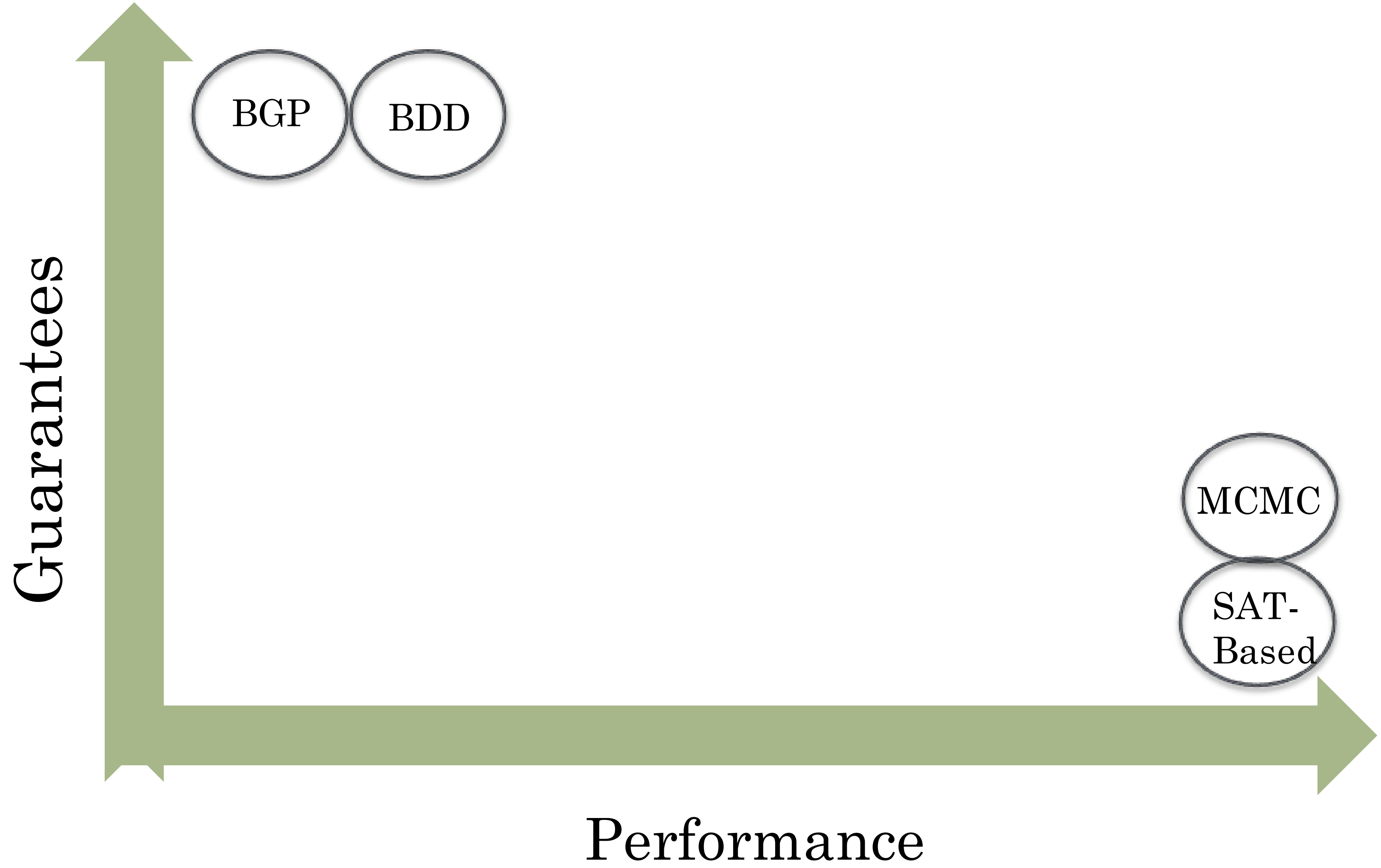
To Recap

- Jerrum, Valiant and Vazirani (1986):
  - Uniform Generator: Polynomial time PTM given access to Σ_2^P oracle
  - Almost-Uniform Generation is inter-reducible to PAC counting

- Bellare, Goldreich, and Petrank (2000)
  - Uniform Generator: Polynomial time PTM given access to NP oracle

Does not work in practice!

Prior Work

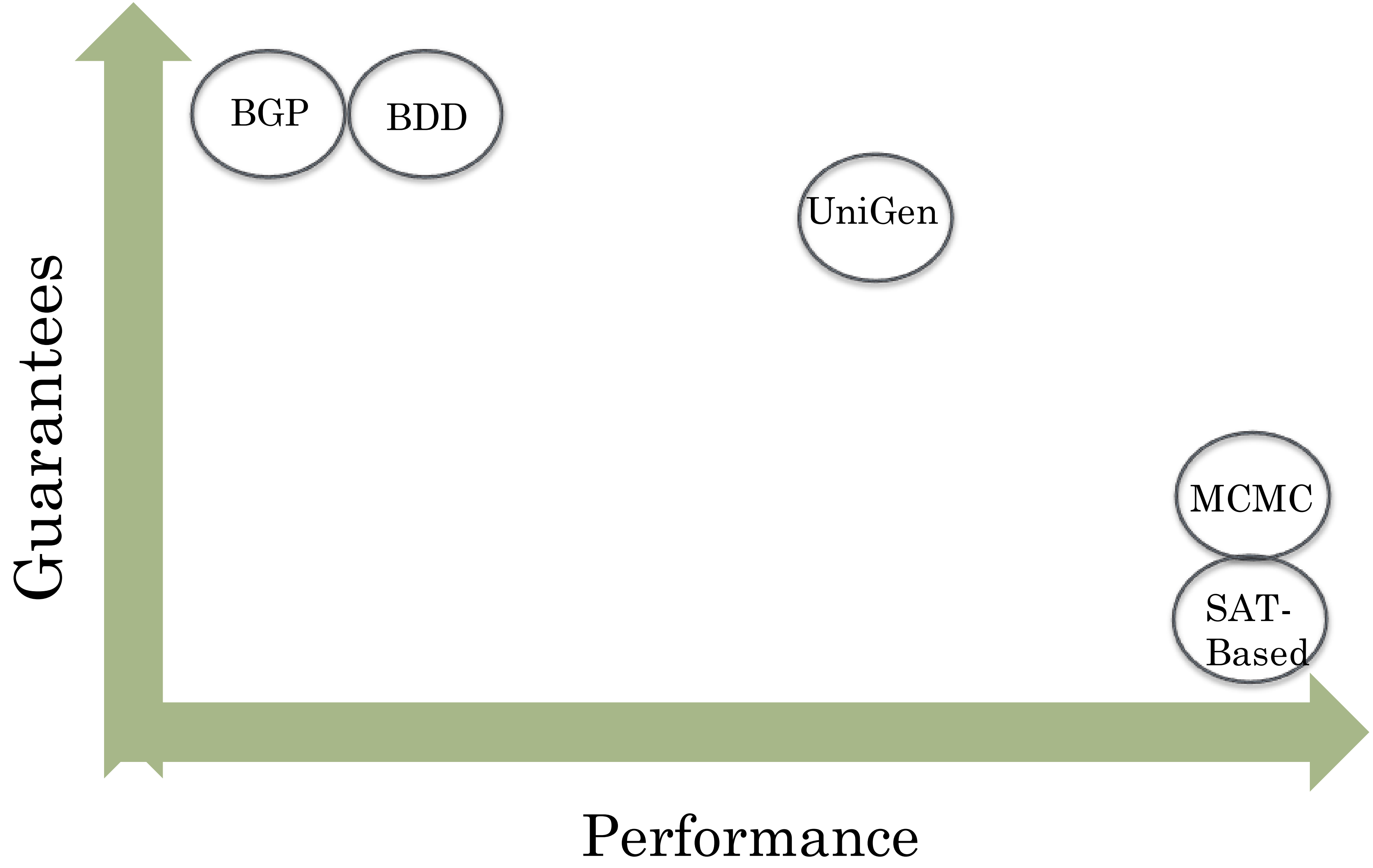


Desires

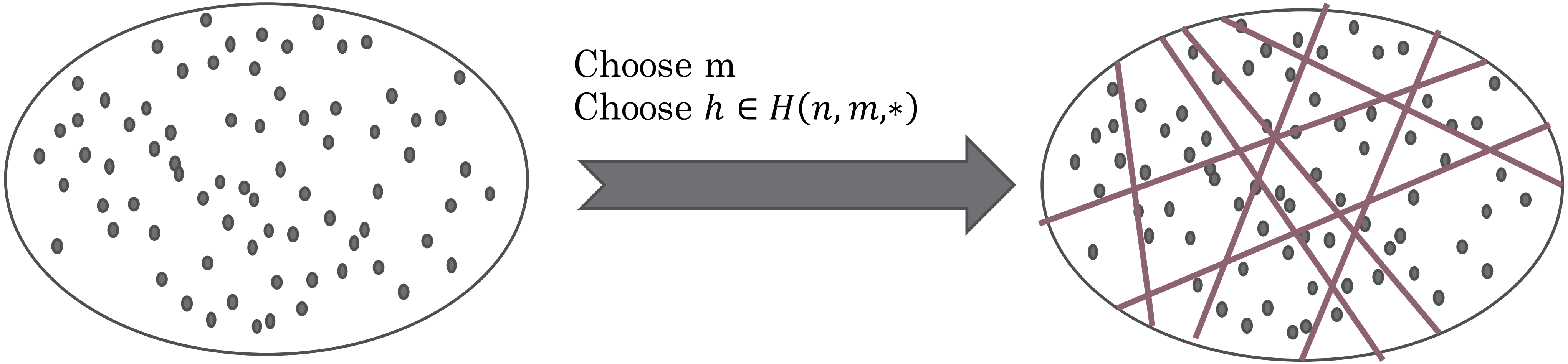
Generator	Relative runtime
State-of-the-art: XORSample'	50000
Ideal Uniform Generator*	10
SAT Solver	1

Experiments over 200+ benchmarks
*: According to EDA experts

Our Contribution



Key Ideas



- For right choice of m, large number of cells are “small”
  - “almost all” the cells are “roughly” equal
- Check if a randomly picked cell is “small”
- If yes, pick a solution randomly from randomly picked cell

Key Challenges

- F: Formula X: Set of variables R_F: Solution space
- $R_{F,h,\alpha}$: Set of solutions for $F \wedge (h(X) = \alpha)$ where
  - $h \in H(n, m,*)$; $\alpha \in \{0,1\}^m$

1. How large is "small" cell ?
2. How much universality do we need?
3. What is the value of m?

Size of cell

q

$$\Pr[\text{ y is output }] = \frac{1}{2^m} * \Pr[\text{Cell is small} \mid \text{ y is in the cell}] * \frac{1}{Size\ of\ cell}$$

Let Size of cell $\in [loThresh, hiThresh]$, Then:

$$\frac{1}{2^m} * \text{q} * \frac{1}{hiThresh} \leq \Pr[y\ is\ output] \leq \frac{1}{2^m} * \text{q} * \frac{1}{loThresh}$$

$$\frac{1}{(1+\varepsilon)|R_F|} \leq \Pr[y \text{ is output }] \leq \frac{(1+\varepsilon)}{|R_F|}$$

$$hiThresh = (1+\varepsilon) * pivot; \quad loThresh = \frac{pivot}{1+\varepsilon}$$

$$pivot = k\left(1+\frac{1}{\varepsilon^2}\right);$$

Losing Independence

Our desire:

$$\Pr\left[\, loThresh \leq |R_{F,h,\alpha}| \leq hiThresh\right] \geq p \ (\geq \frac{1}{2})$$

$$\Pr\left[\frac{pivot}{1+\varepsilon} \leq |R_{F,h,\alpha}| \leq (1+\varepsilon)pivot\right] \geq p \ (\geq \frac{1}{2})$$

Suppose $h \in H(n,m,*)$ *and* $m = \log \frac{|R_F|}{pivot}$

Then, $E[|R_{F,h,\alpha}|] = \frac{|R_F|}{2^m} = pivot$

Concentration bound $\Rightarrow$ k-universal (small constant)

How many cells?

- Our desire: $m = \log \frac{|R_F|}{pivot}$
  - But determining $|R_F|$ is expensive (#P complete)

- How about approximation?
  - $ApproxMC(F, \varepsilon, \delta)$ returns C:

$$\Pr\left[\frac{|R_F|}{1+\varepsilon} \leq C \leq (1+\varepsilon)|R_F|\right] \geq 1-\delta$$

  - $q = \log C \; - \log pivot$

  - Concentrate on m = q-1, q, q+1

UniGen(F,ε)

1. C = ApproxMC(F,ε) One time execution
2. Compute pivot, loThresh, hiThresh
3. $q = \log|C| - \log pivot$
4. for i in {q-1, q, q+1}:
5. Choose h **randomly*** from H(n,i,3)
6. Choose α randomly from $\{0,1\}^m$
7. If $(loThresh \leq |R_{F,h,\alpha}| \leq hiThresh)$:
8. Pick $y \in R_{F,h,\alpha}$ randomly

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ Run for every sample required

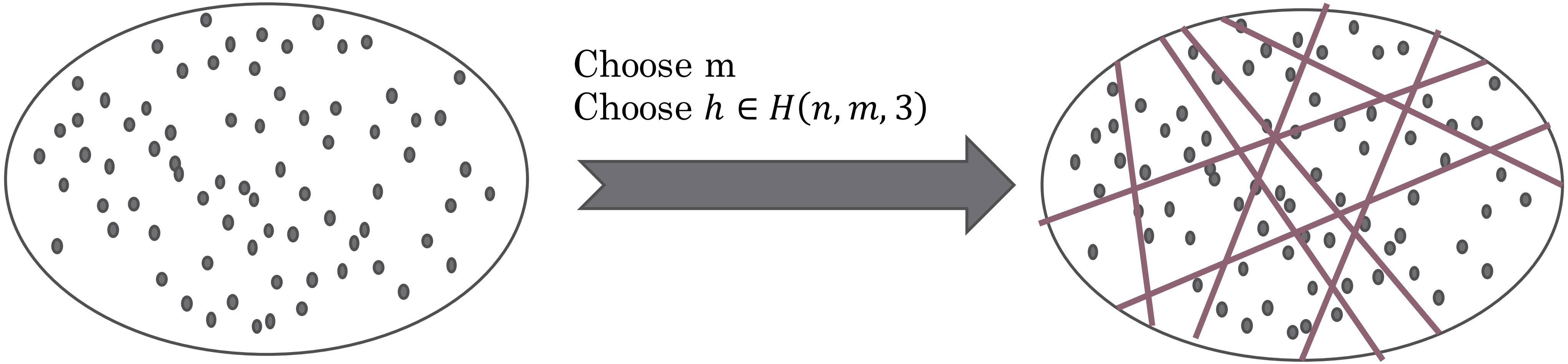
Are we back to JVV?

NOT Really

- JVV makes linear (in n) calls to Approximate counter compared to just 1 in UniGen

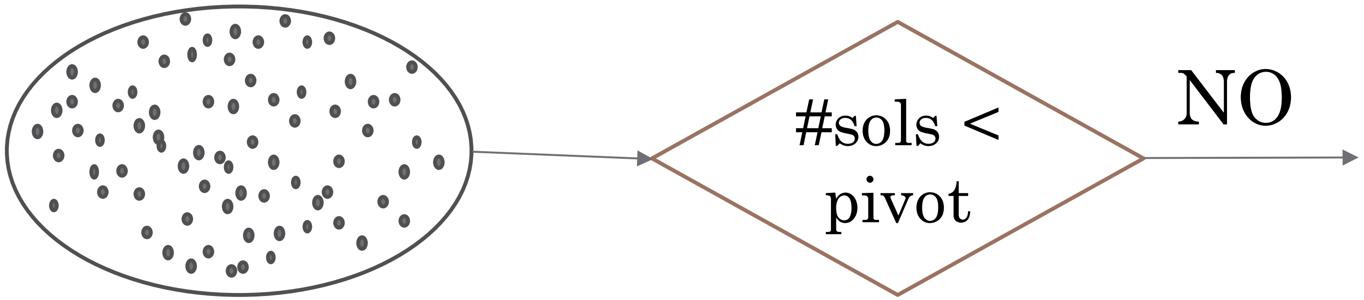
- # of calls to ApproxMC is only 1 regardless of the number of samples required unlike JVV

PAC Counter: ApproxMC(F,ε, δ)



- For right choice of m, large number of cells are "small"
  - "almost all" the cells are "roughly" equal
- Check if a randomly picked cell is "small"
- If yes, then estimate = # of solutions in cell * 2^m

ApproxMC(F,ε, δ)



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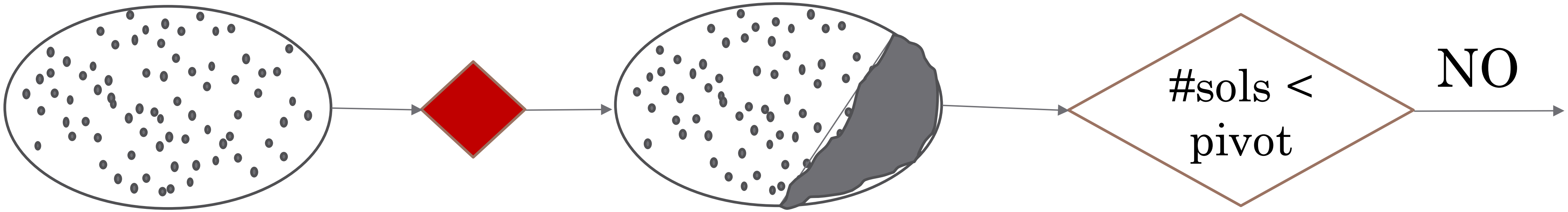

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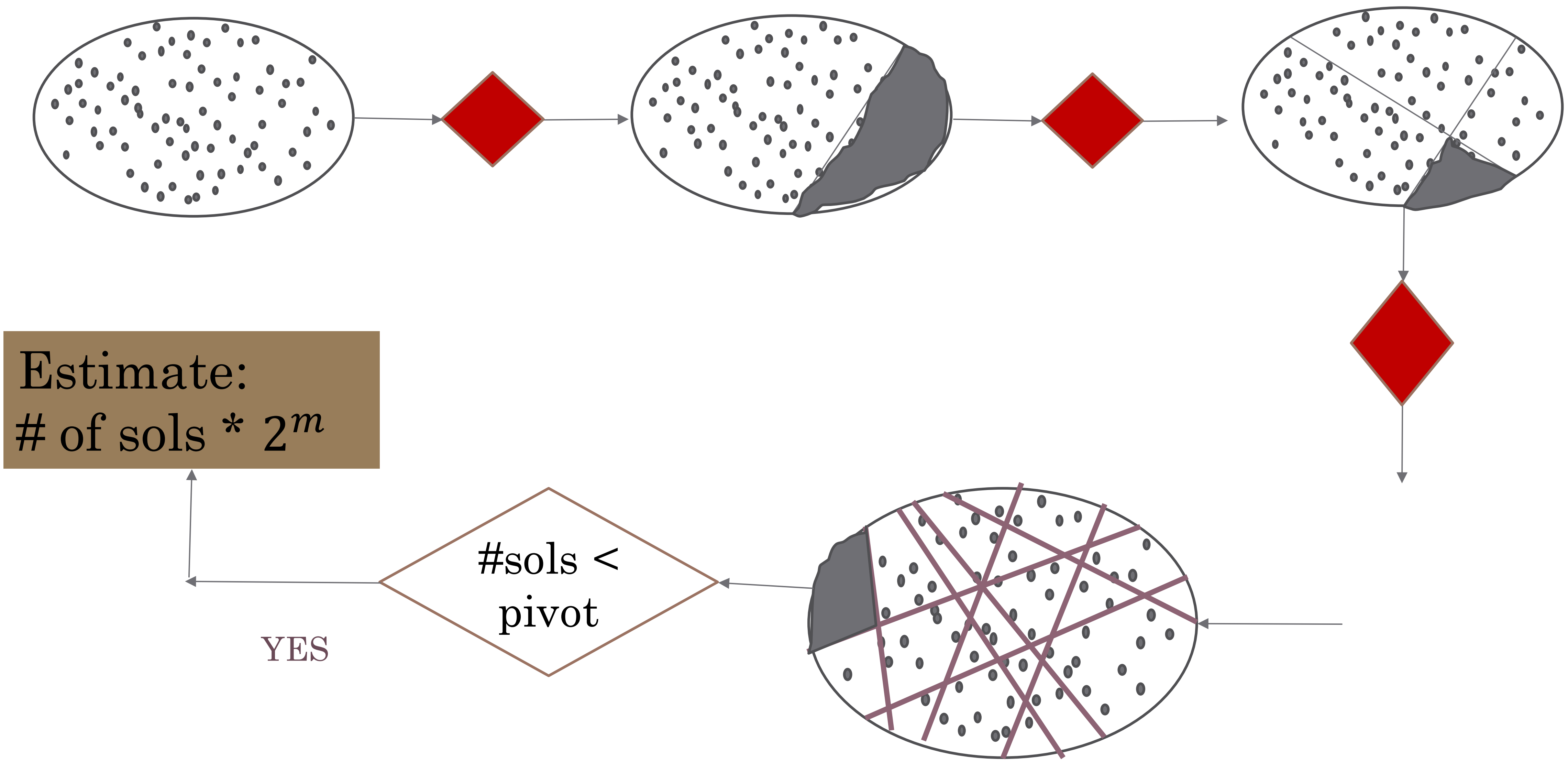


ApproxMC(F, ε, δ)

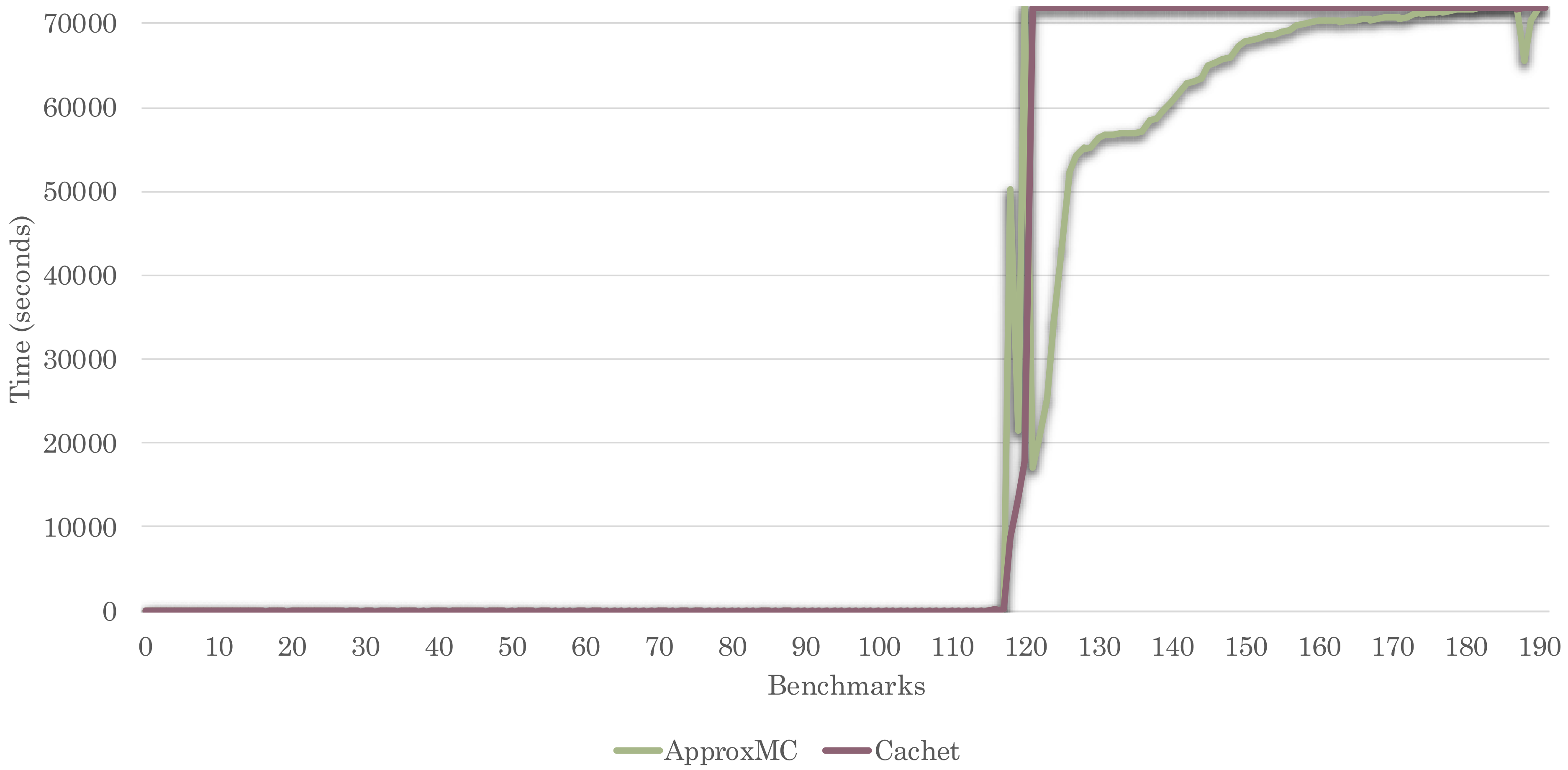
Key Lemmas

Let $m^* = \log|R_F| - \log pivot$

Lemma 1: The algorithm terminates with $m \in [m^* - 1, m^*]$ with high probability

Lemma 2: The estimate from a randomly picked cell for $m \in [m^* - 1, m^*]$ is correct with high probability

Results: Performance Comparison



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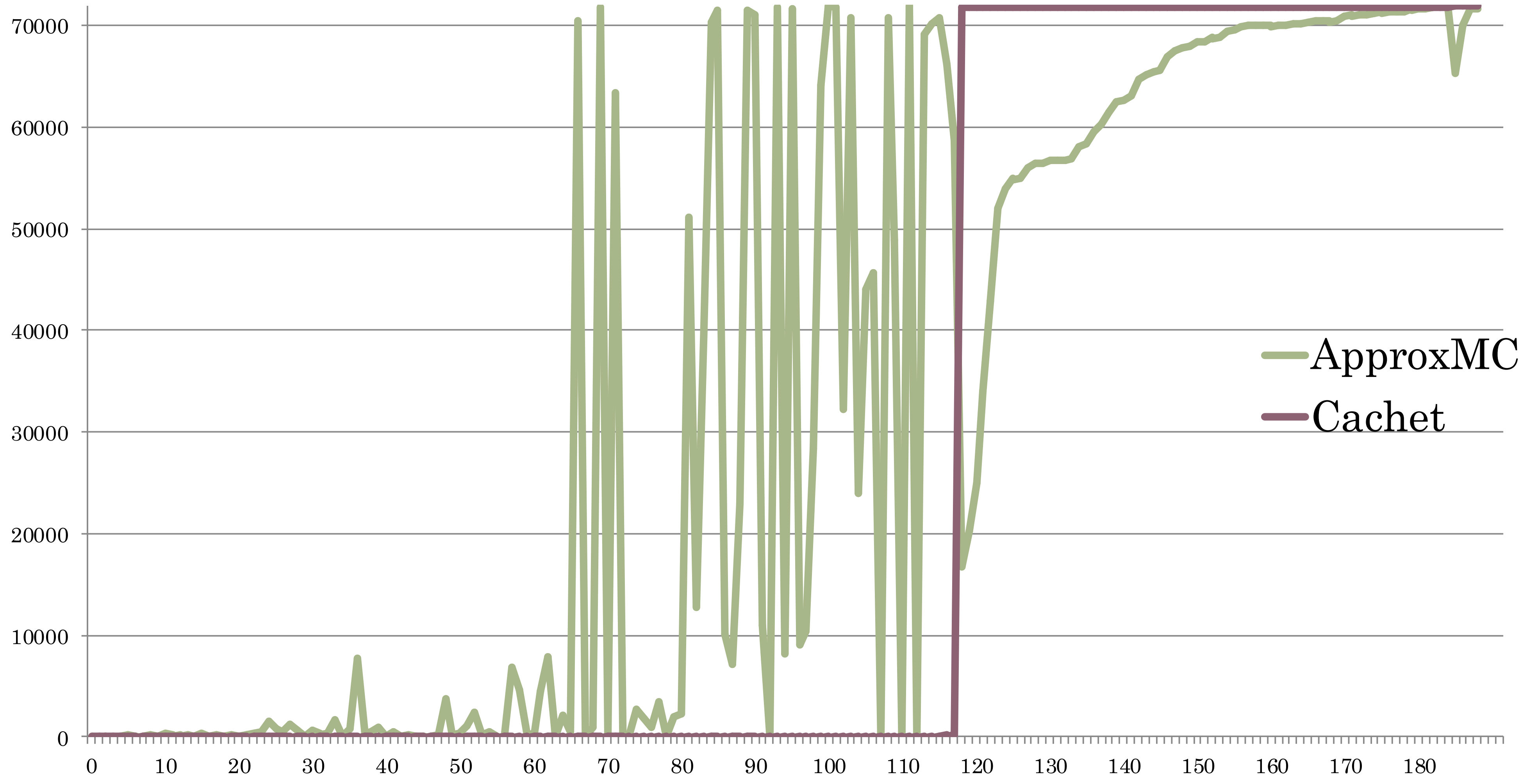
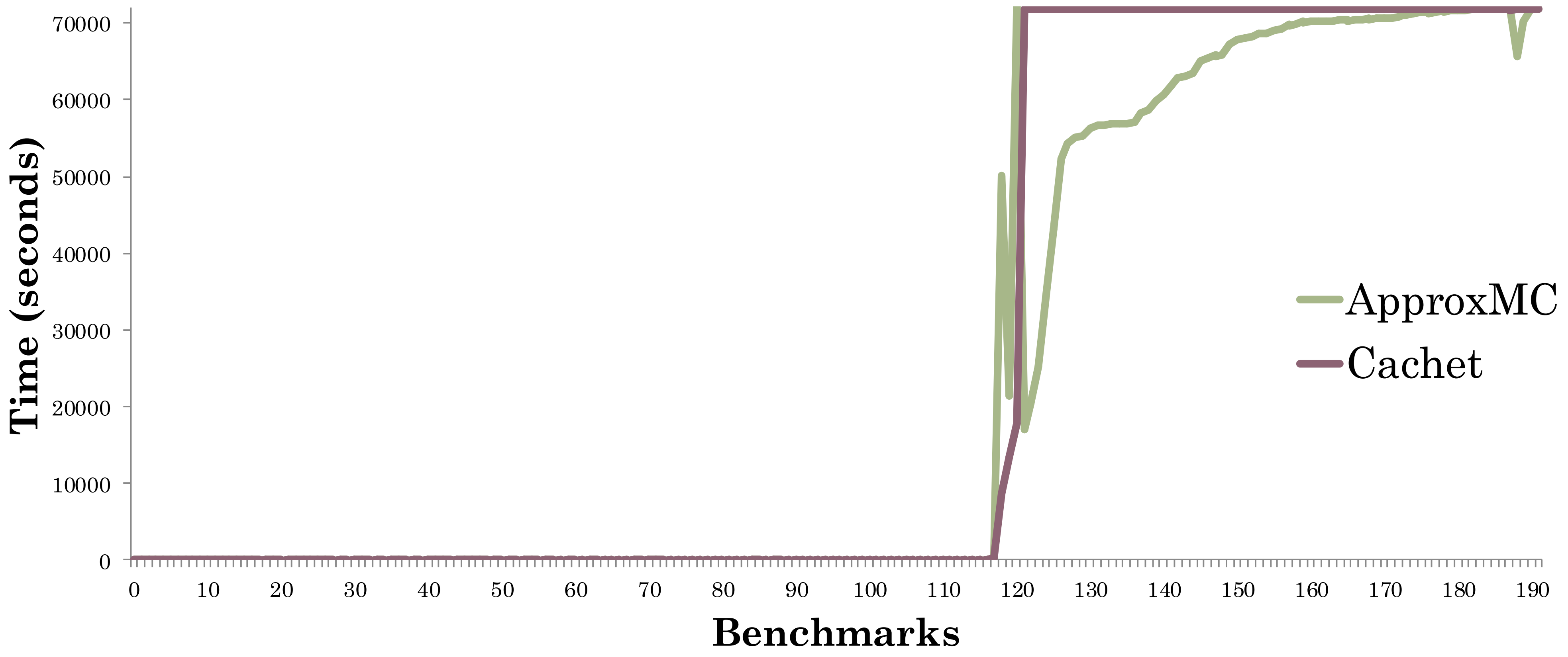
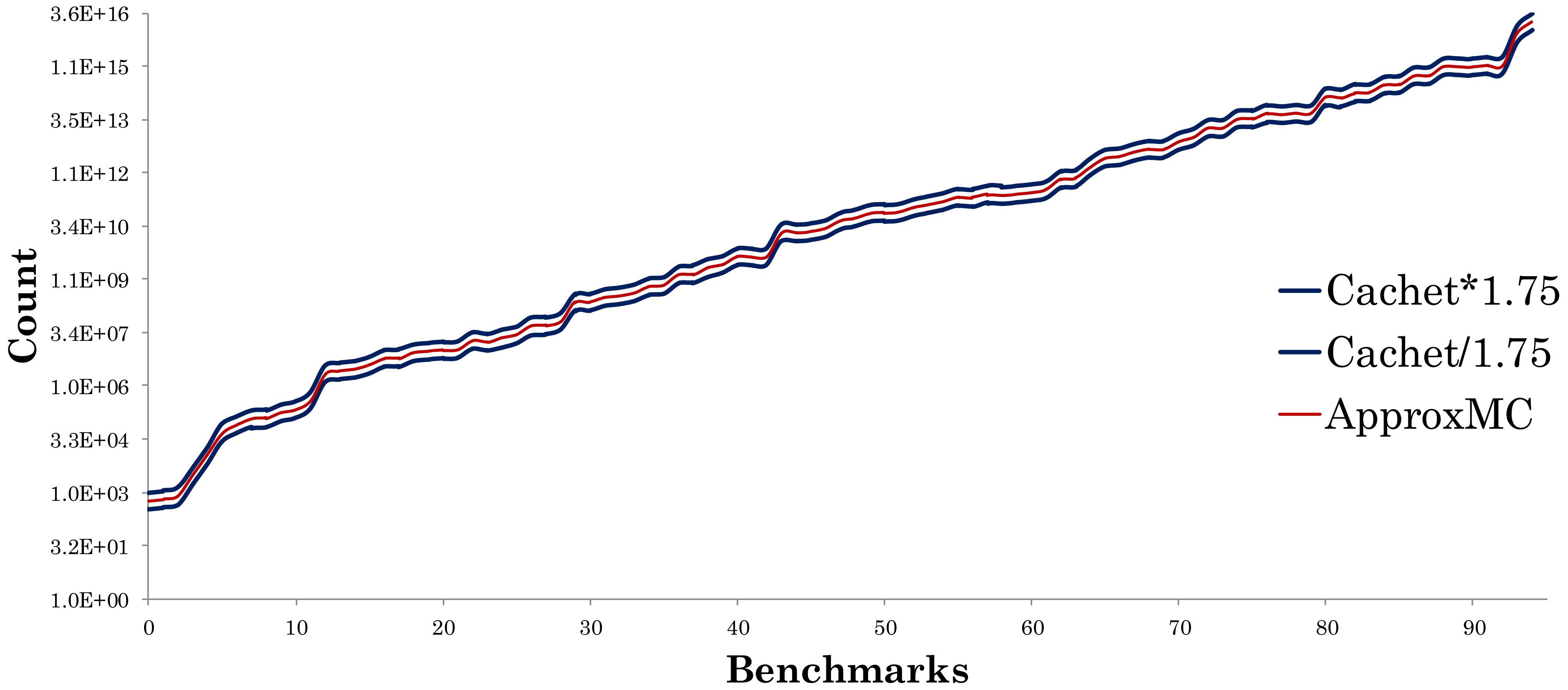


Can Solve a Large Class of Problems



Large class of problems that lie beyond the exact counters but can be computed by ApproxMC

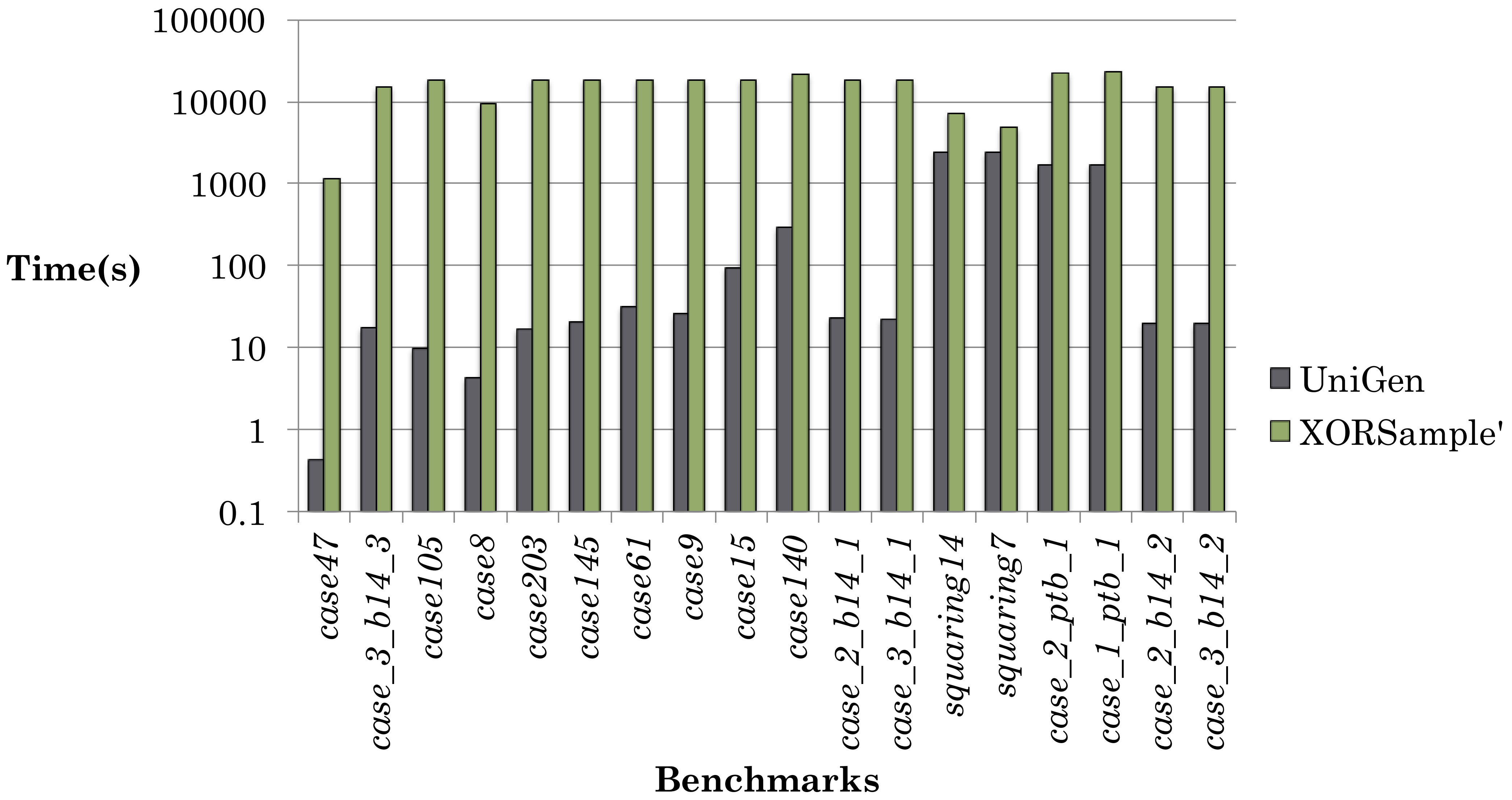
Mean Error: Only 4% (allowed: 75%)



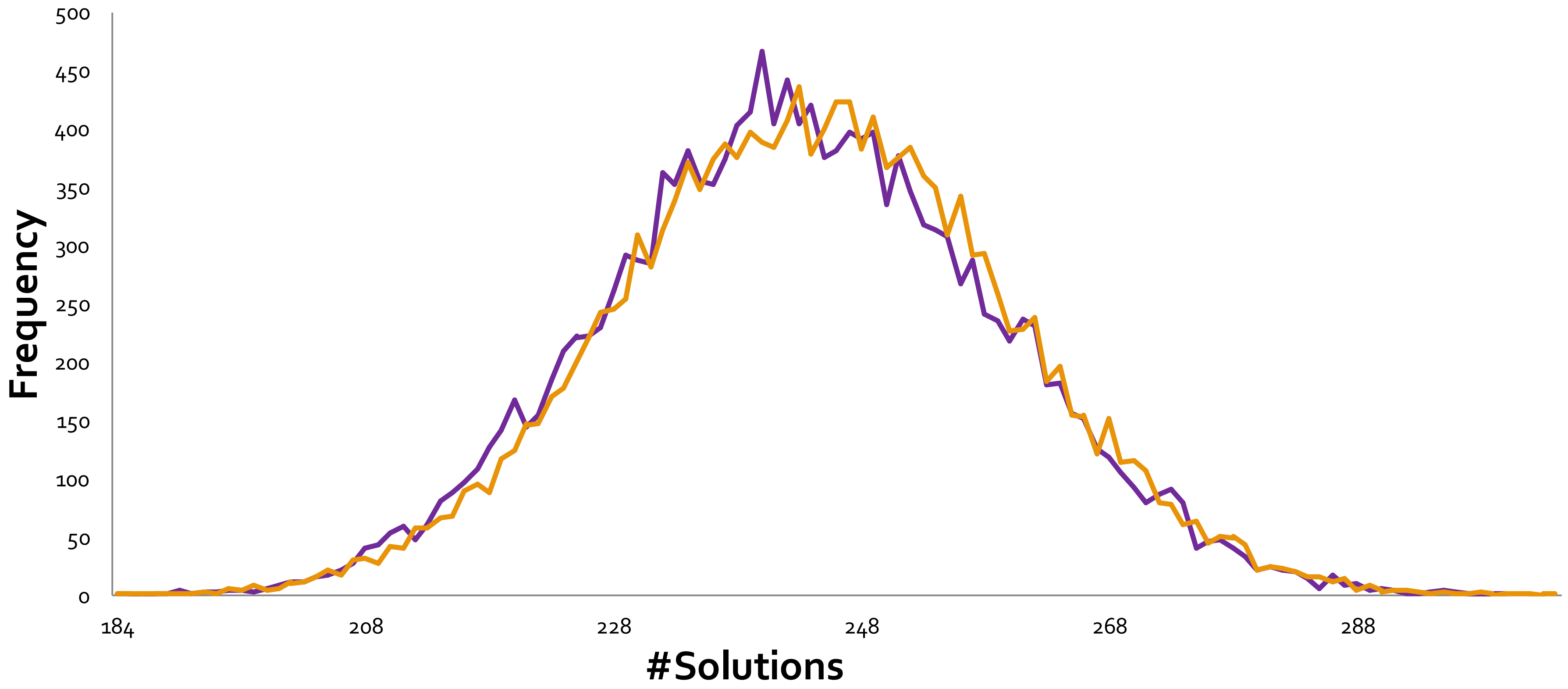
Mean error: 4% – much smaller than the theoretical guarantee of 75%

Runtime Performance of UniGen

1-2 Orders of Magnitude Faster

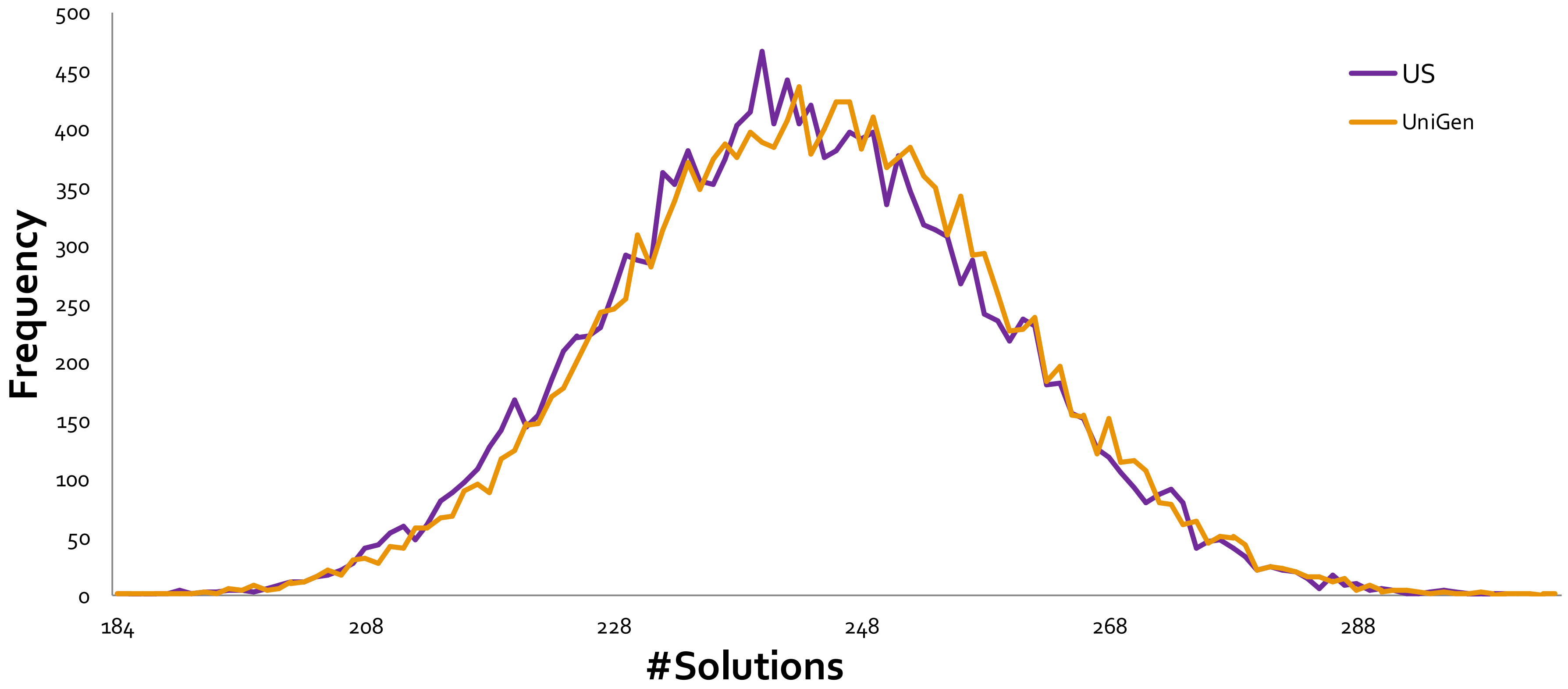


Results: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: $4x10^6$; Total Solutions : 16384

Results: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: $4\text{x}10^6$; Total Solutions : 16384

So far

- The first scalable approximate model counter
- The first scalable uniform generator
- Outperforms state-of-the-art generators/counters

Are we done?

Where are we?

Generator	Relative runtime
State-of-the-art: XORSample'	50000
UniGen	~5000
Ideal Uniform Generator*	10
SAT Solver	1

Experiments over 200+ benchmarks
*: According to EDA experts

XOR-Based Hashing

- Partition 2^n space into 2^m cells
- Variables: X_1, X_2, X_3,….., X_n
- Pick every variable with prob. ½ ,XOR them and add 0/1 with prob. ½
- $X_1+X_3+X_6+\ldots X_{n-1} + 0$
- To construct h: $\{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- $\alpha \in \{0,1\}^m \rightarrow$ Set every XOR equation to 0 or 1 randomly
- The cell: F $\wedge$ XOR (CNF+XOR)

XOR-Based Hashing

- **CryptoMiniSAT**: Efficient for CNF+XOR
- Avg Length : n/2
- Smaller XORs → better performance

How to shorten XOR clauses?

Independent Support

- Set I of variables such that assignments to these uniquely determine assignments to rest of variables (for satisfying assignments)
- If s_1 and s_2 agree on I then $s_1 = s_2$
- c ↔ (a V b) ; Independent Support I: {a, b}
- ***Key Idea***: Hash only on the independent variables

Independent Support

- Hash only on the Independent Support
- Average size of XOR: $n/2$ to $|I|/2$

Formal Definition

Input Formula: F, Solution space: R_F

$\forall \sigma_1, \sigma_2 \in R_F$, If σ_1 and σ_2 agree on I, then $\sigma_1 = \sigma_2$

$$F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i \mid x_i \in I} (x_i = y_i) \implies \bigwedge_j (x_j = y_j)$$

$$\text{where } F(y_1, \ldots, y_n) = F(x_1 \to y_1, \ldots, x_n \to y_n)$$

Minimal Unsatisfiable Subset

- Given $\Psi = H_1 \wedge H_2 \cdots H_m$
  - Find subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik} \wedge \Omega$ is UNSAT

    Unsatisfiable subset

  - Find **minimal** subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik}$ is UNSAT

    Minimal Unsatisfiable subset

Key Idea

$$F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i | x_i \in I} (x_i = y_i) \implies \bigwedge_j (x_j = y_j)$$

$$Q_{F,I} = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i | x_i \in I} (x_i = y_i) \wedge \neg \left(\bigwedge_j (x_j = y_j) \right).$$

Theorem: $Q_{F,I}$ is unsatisfiable if and only if I is independent support

Key Idea

$$H_1 = \{x_1 = y_1\},\ \ldots,\ H_n = \{x_n = y_n\}$$

$$\Omega = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge (\neg \bigwedge_j (x_j = y_j))$$

$I = \{x_i\}$ is Independent Support iff $H^I \wedge \Omega$ is unsatisfiable where $H^I = \{H_i \mid x_i \in I\}$

Group-Oriented Minimal Unsatisfiable Subset

- Given $\Psi = H_1 \wedge H_2 \cdots H_m \wedge \Omega$
  - Find subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik} \wedge \Omega$ is UNSAT

    Group Oriented Unsatisfiable subset

  - Find **minimal** subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik} \wedge \Omega$ is UNSAT

    Group Oriented Minimal Unsatisfiable subset

Minimal Independent Support

$$H_1 = \{x_1 = y_1\},\ \ldots,\ H_n = \{x_n = y_n\}$$

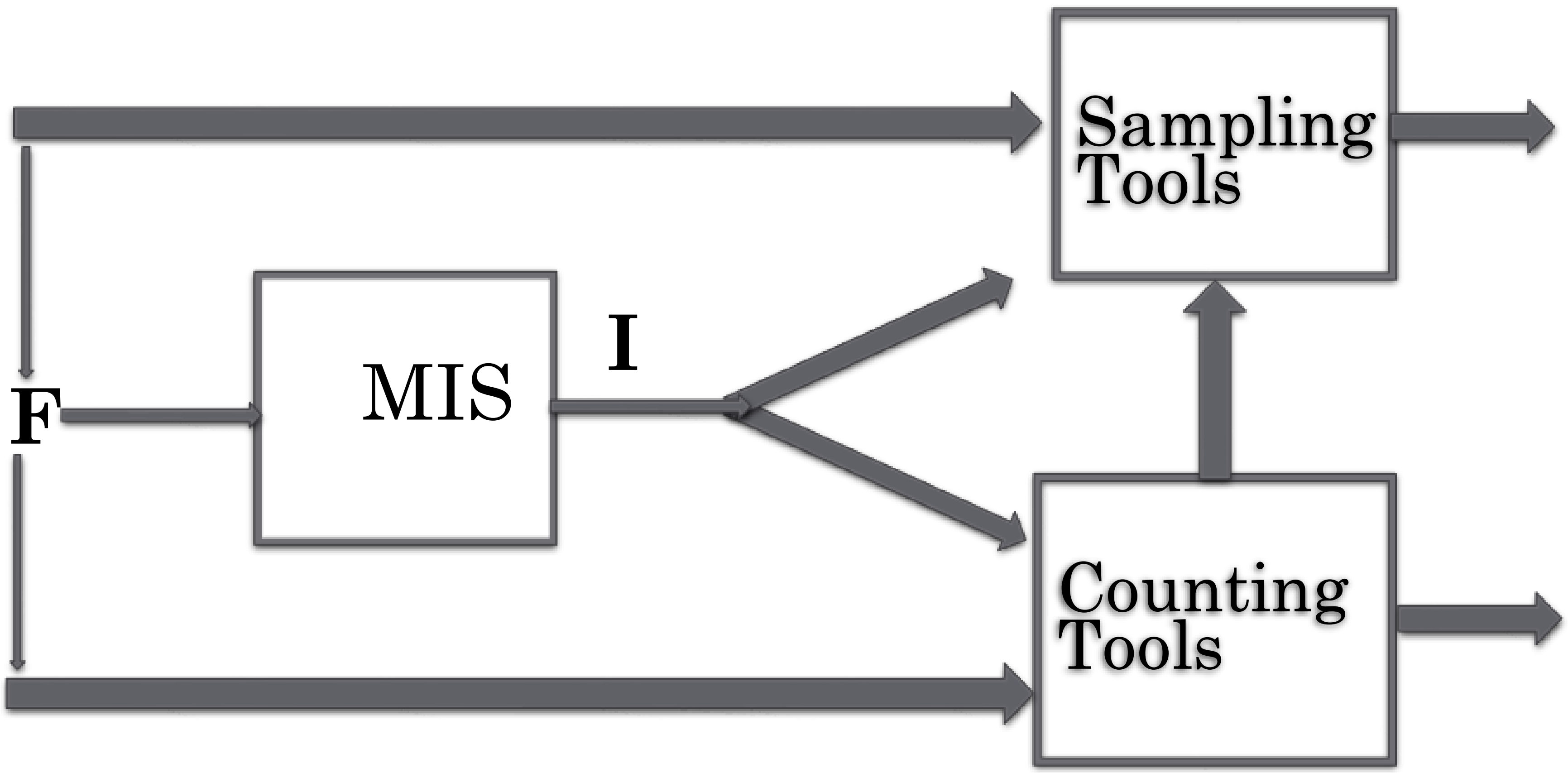
$$\Omega = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge (\neg \bigwedge_j (x_j = y_j))$$

$I = \{x_i\}$ is minimal Independent Support iff H^I is minimal unsatisfiable subset where $H^I = \{H_i \mid x_i \in I\}$

Key Idea

Minimal Independent Support (MIS) → Minimal Unsatisfiable Subset (MUS)

Impact on Sampling and Counting Techniques

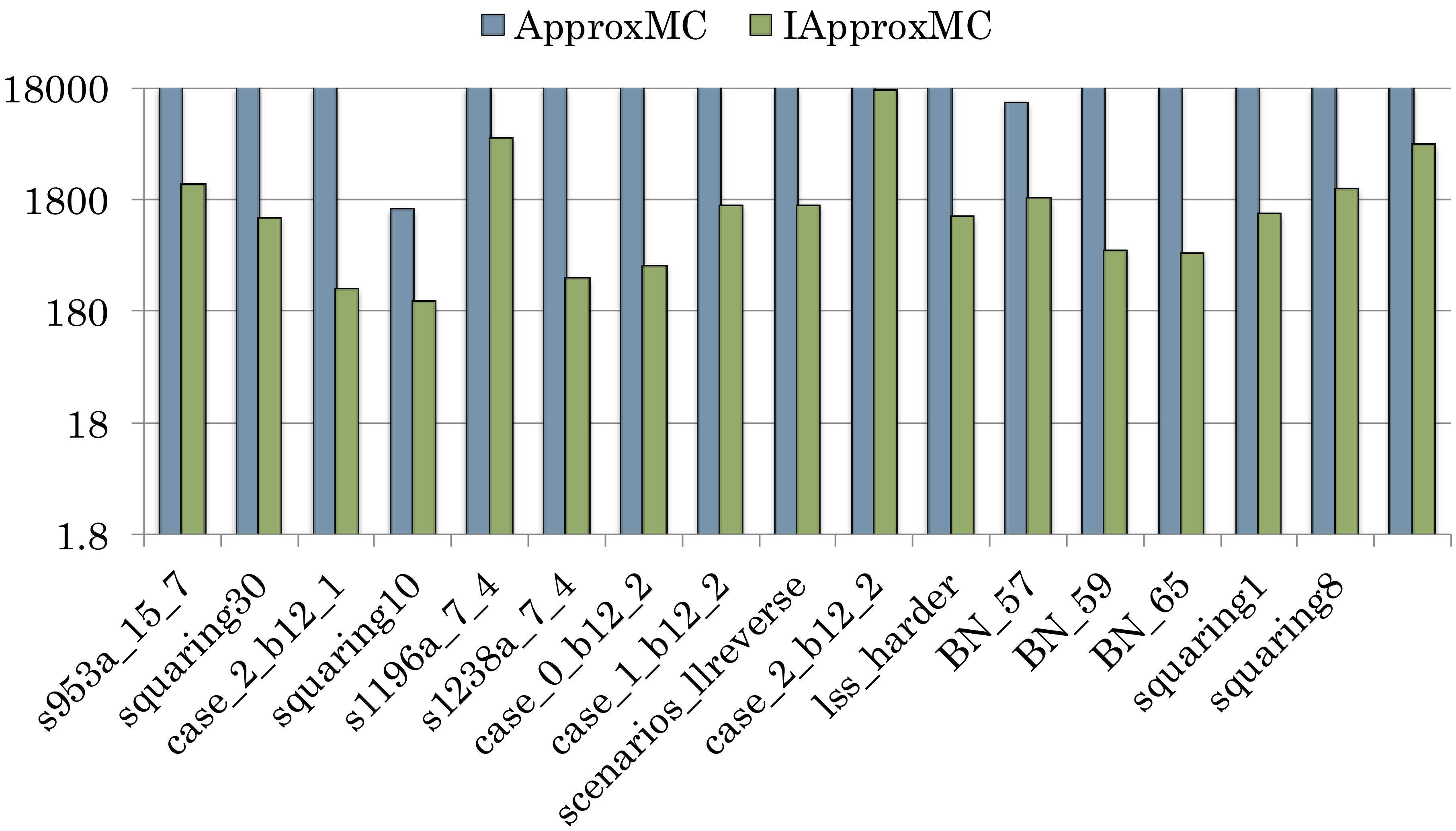


What about complexity

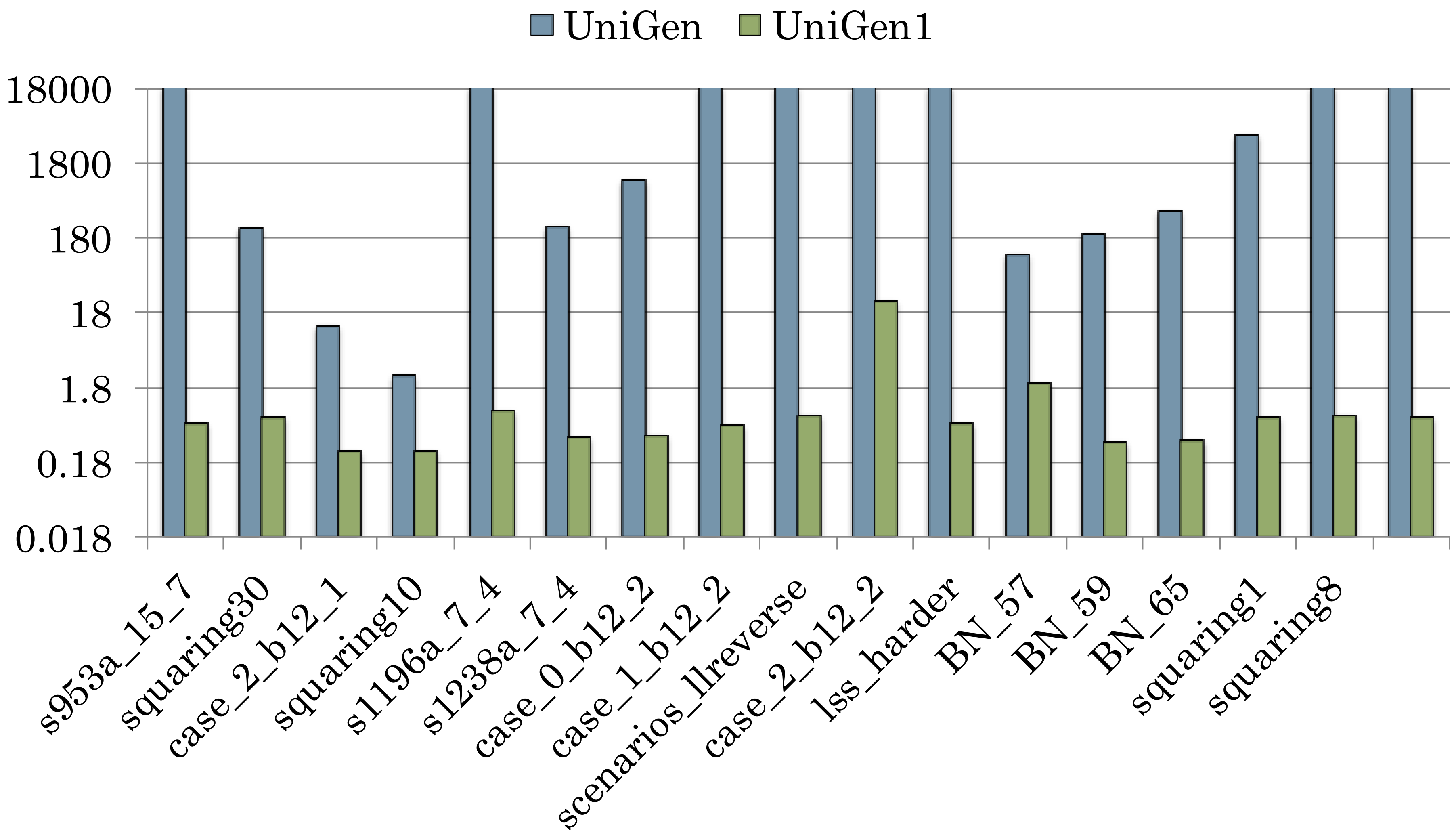
- Computation of MUS: FP^{NP}
- Why solve a FP^{NP} for almost-uniform generation/approximate counter (PTIME PTM with NP Oracle)

Settling the debate through practice!

Performance Impact on Approximate Model Counting



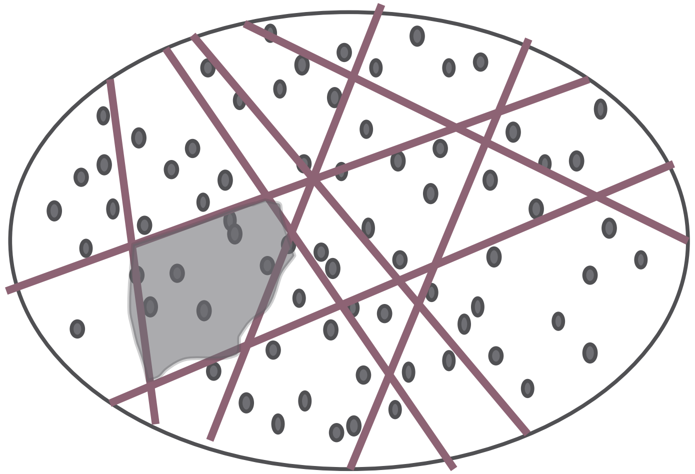
Performance Impact on Uniform Sampling



Where are we?

Generator	Relative runtime
State-of-the-art: XORSample'	50000
UniGen	5000
UniGen1	470
Ideal Uniform Generator*	10
SAT Solver	1

Back to basics



\# of solutions in "small" cell $\in [loThresh, hiThresh]$
We pick one solution
"Wastage" of loThresh solutions

Pick *loThresh* samples!

3-Universal and Independence of Samples

3-Universal hash functions:

- Choose hash function randomly
- For arbitrary distribution on solutions=> All cells are *roughly* equal in expectation

- But:
  - While each input is hashed **uniformly**
  - And each 3-solutions set is hashed **independently**
  - A 4-solutions set ***might not*** be hashed **independently**

Balancing Independence

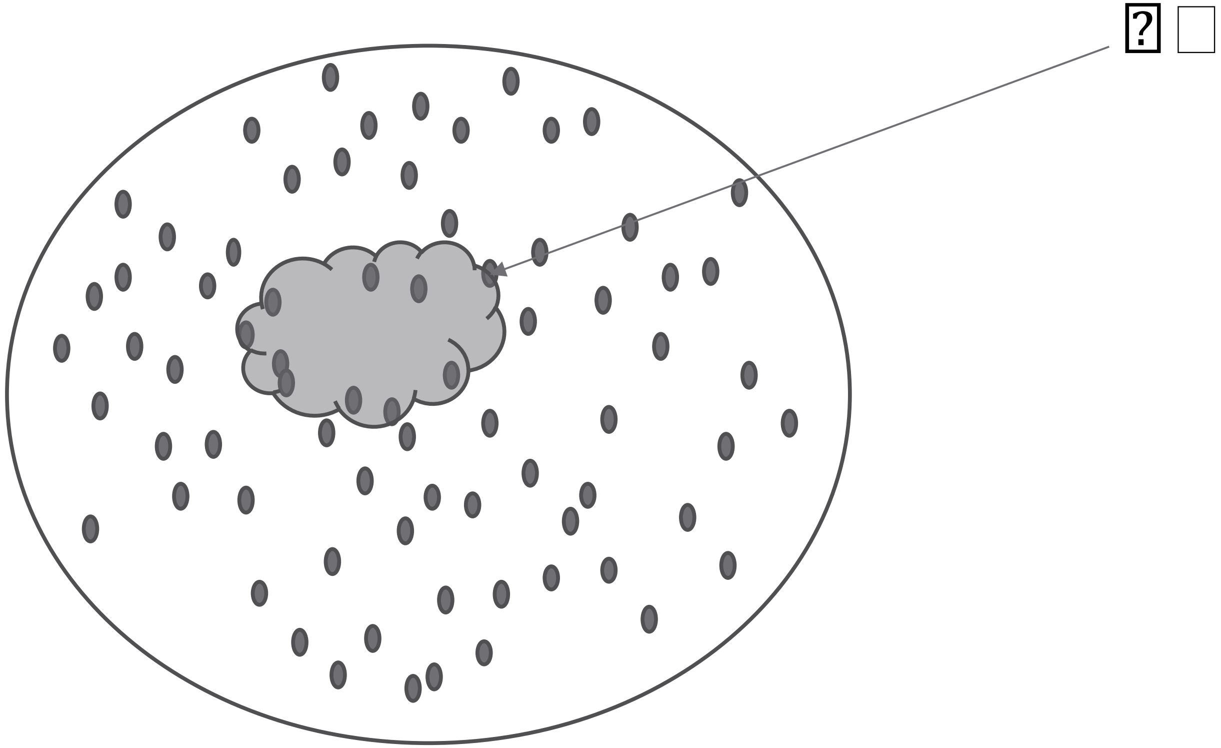
For $h \in H(n, m, 3)$

- Choosing up to 3 samples => Full independence among samples

- Choosing loThresh (>> 3) samples => Loss of independence

Why care about Independence

Convergence requires multiplication of probabilities



If every sample is independent => Faster convergence

The principle of principled compromise!

- Choosing up to 3 samples => Full independence among samples
- Choosing loThresh (>> 3) samples => Loss of independence
  - “Almost-Independence” among samples
  - Still provides strong theoretical guarantees of coverage

Strong Guarantees

- $$L = \#\ of\ samples < |R_F|$$

$$\frac{L}{(1+\varepsilon)|R_F|} \leq \Pr[\text{y is output}] \leq 1.02(1+\varepsilon)\frac{L}{|R_F|}$$

- ~~Polynomial~~ Constant number of SAT calls per sample
  - After one call to ApproxMC

Bug-finding effectiveness

$$\text{bug frequency f} = \frac{|B|}{|R_F|}$$

	UniGen	UniGen2
relative number of SAT calls	$\frac{3 \cdot hiThresh(1+\nu)(1+\varepsilon)}{0.52}$	$\frac{3 \cdot hiThresh}{0.62 \cdot loThresh} \frac{(1+\widehat{\nu})(1+\varepsilon)}{1-\widehat{\nu}}$

Simply put,
#of SAT calls for UniGen2 << # of SAT calls for UniGen

Bug-finding effectiveness

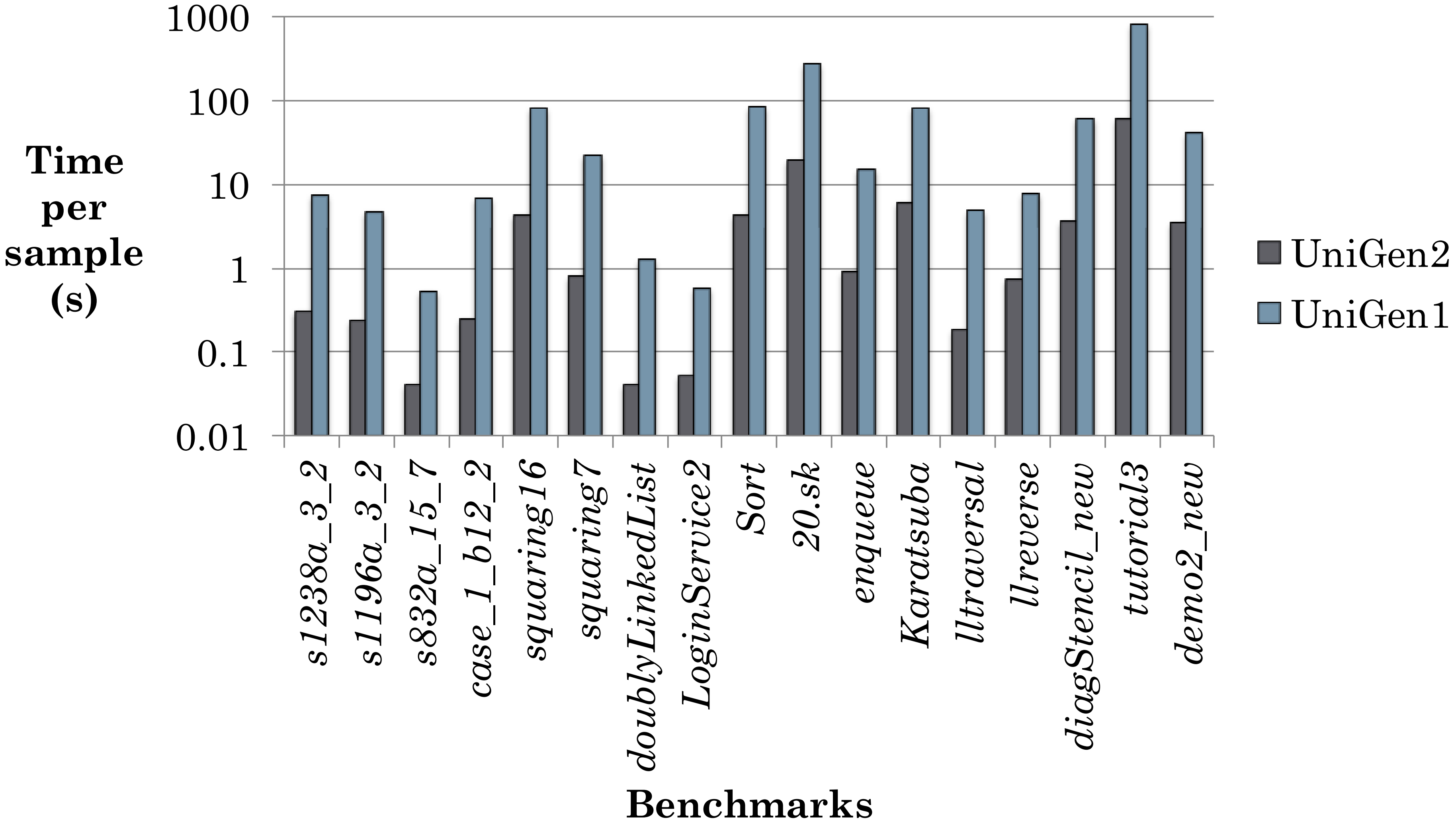
bug frequency $f = 1/10^4$

find bug with probability $\geq 1/2$

	UniGen	UniGen2
Expected number of SAT calls	4.35×10^7	3.38×10^6

An order of magnitude difference!

~20 times faster than UniGen1



Where are we?

Generator	Relative runtime
State-of-the-art: XORSample'	50000
UniGen	5000
UniGen1	470
UniGen2	20
Ideal Uniform Generator*	10
SAT Solver	1

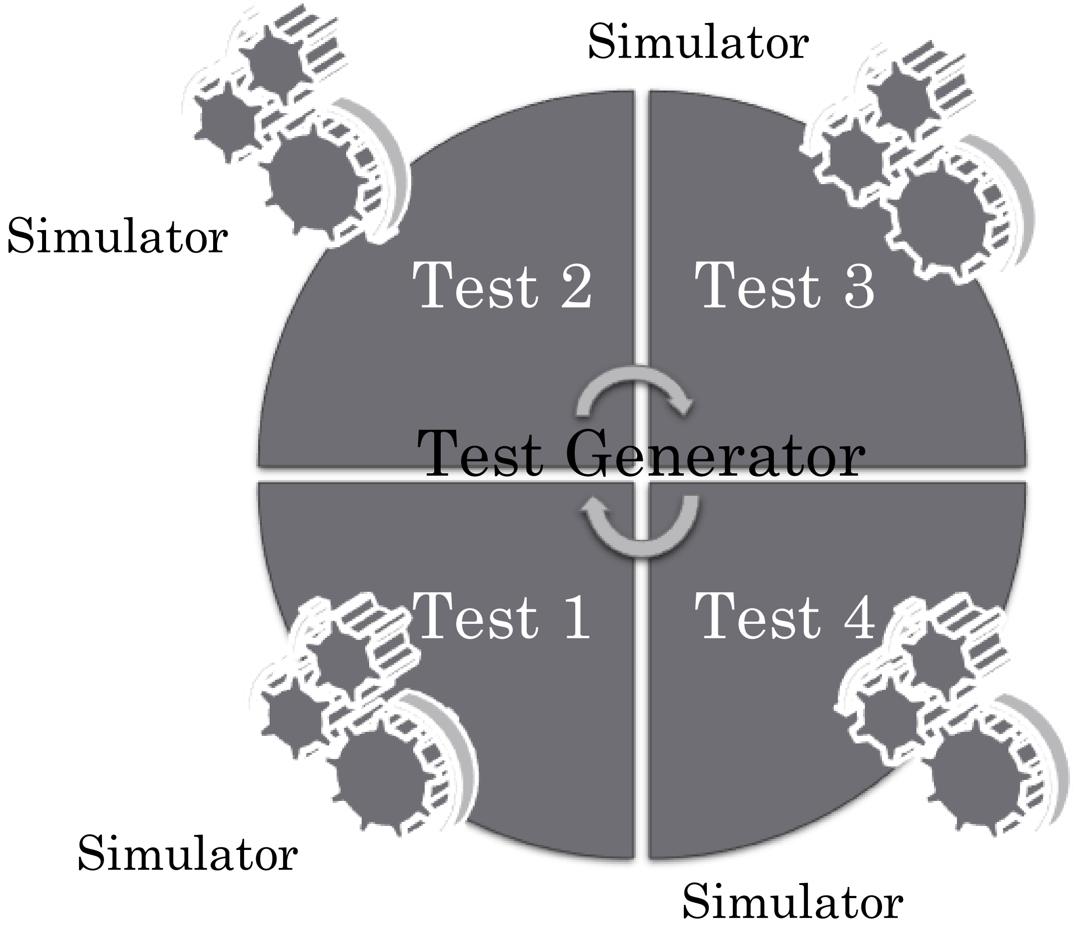
The Final Push....

- UniGen requires one time computation of ApproxMC
- Generation of samples in fully distributed fashion

(Previous algorithms lacked the above property)

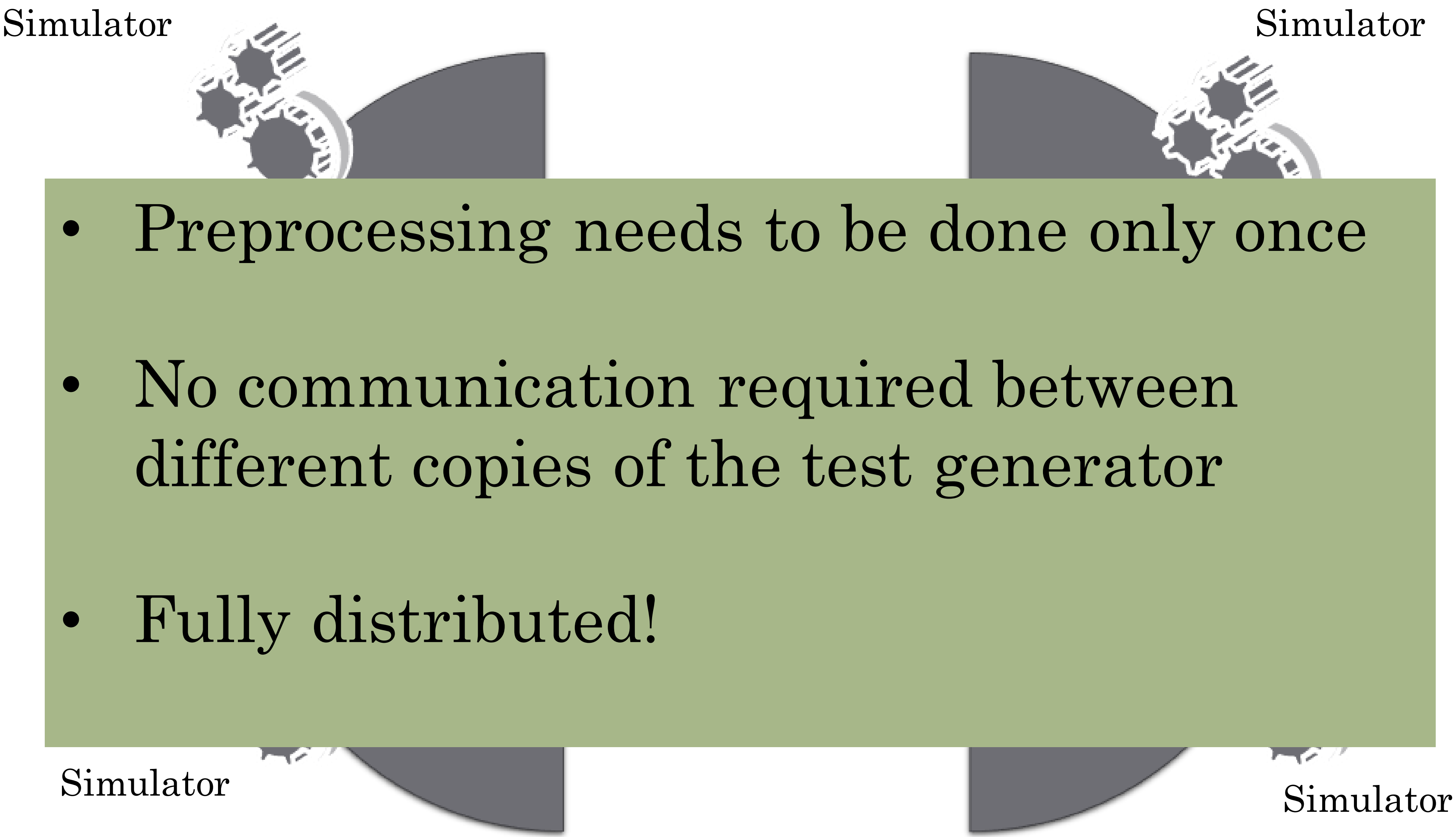
- New paradigms!

Current Paradigm of Simulation-based Verification



- Can not be parallelized since test generators maintain "global state"
- Loses theoretical guarantees (if any) of uniformity

New Paradigm of Simulation-based Verification

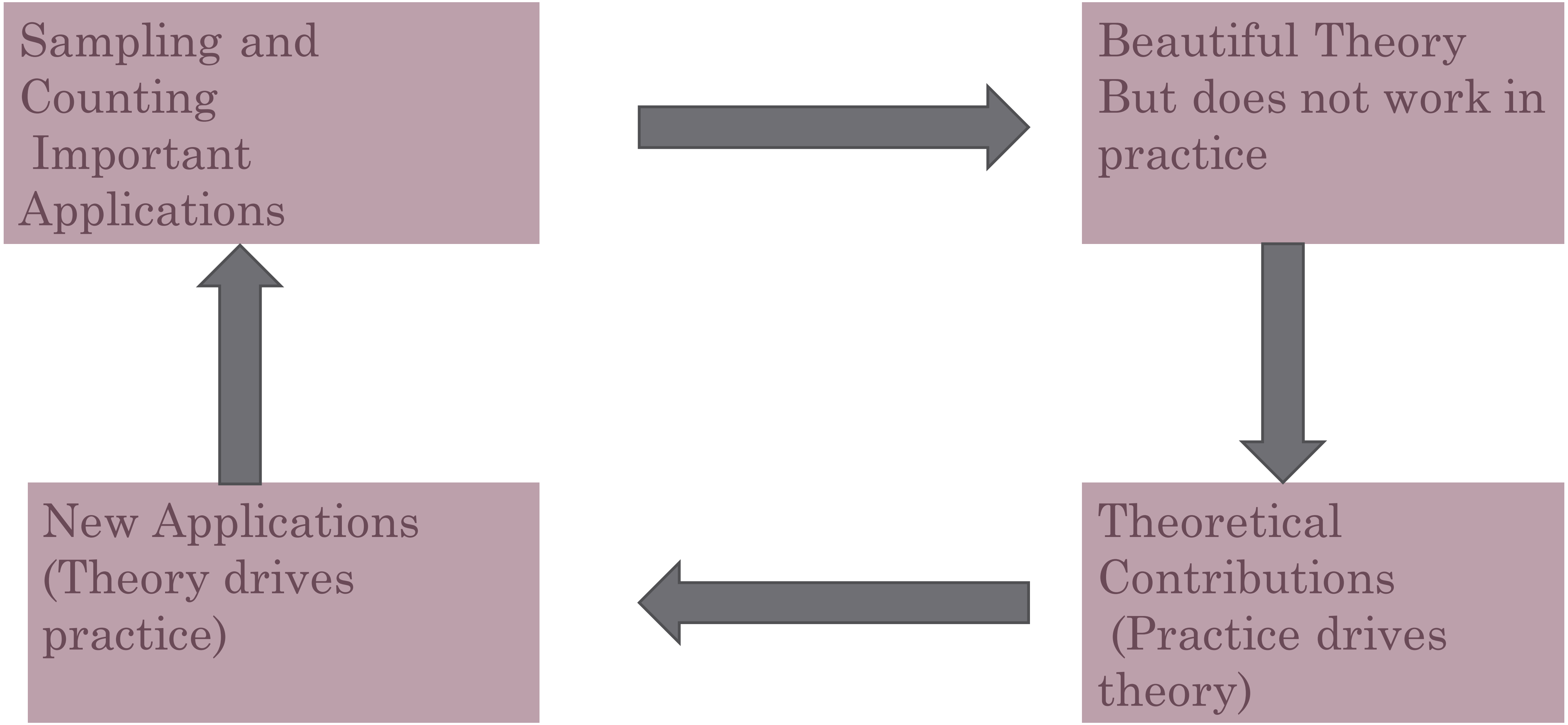


- Preprocessing needs to be done only once
- No communication required between different copies of the test generator
- Fully distributed!

Closing in…

Generator	Relative runtime
State-of-the-art: XORSample’	50000
UniGen	5000
UniGen1	470
UniGen2	20
Multi-core UniGen2	10 (two cores)
Ideal Uniform Generator*	10
SAT Solver	1

So what happened….



Future Directions

Extension to More Expressive domains

- Efficient hashing schemes
  - Extending bit-wise XOR to richer constraint domains provides guarantees but no advantage of SMT progress

- Solvers to handle F + Hash efficiently
  - CryptoMiniSAT has fueled progress for SAT domain
  - Similar solvers for other domains?

Handling Distributions

- Given: CNF formula F and Weight function W over assignments
- Weighted Counting: sum the weight of solutions
- Weighted Sampling: Sample according to weight of solution
- Wide range of applications in Machine Learning
- Extending universal hashing works only in theory so far