# CS344: Introduction to Artificial Intelligence (associated lab: CS386)

Pushpak Bhattacharyya
CSE Dept.,
IIT Bombay

Lecture 23: Perceptrons and their computing power

8th March, 2011

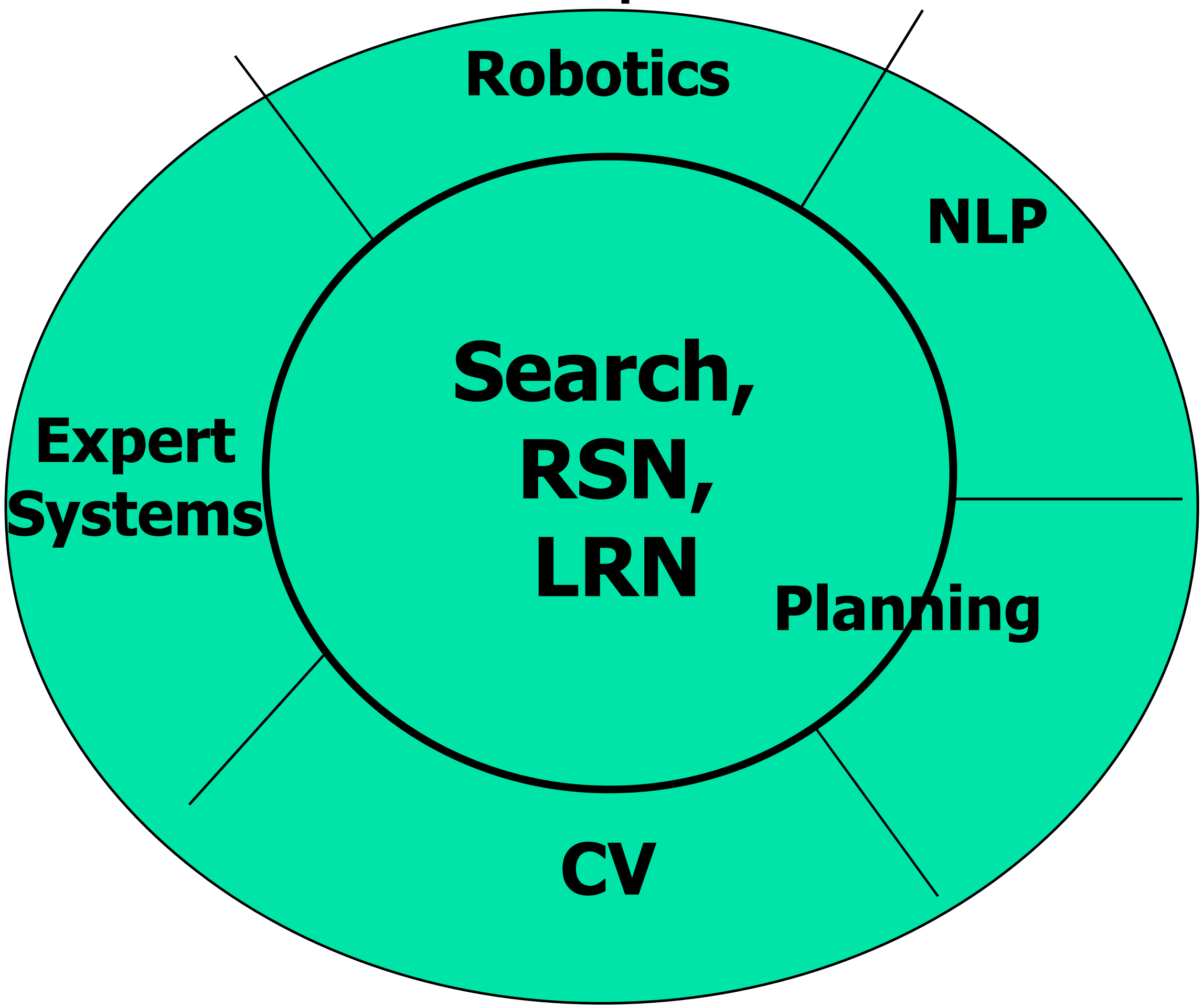
*(Lectures 21 and 22 were on Text Entailment by Prasad Joshi)*

**A perspective of AI**

**Artificial Intelligence - Knowledge based computing**

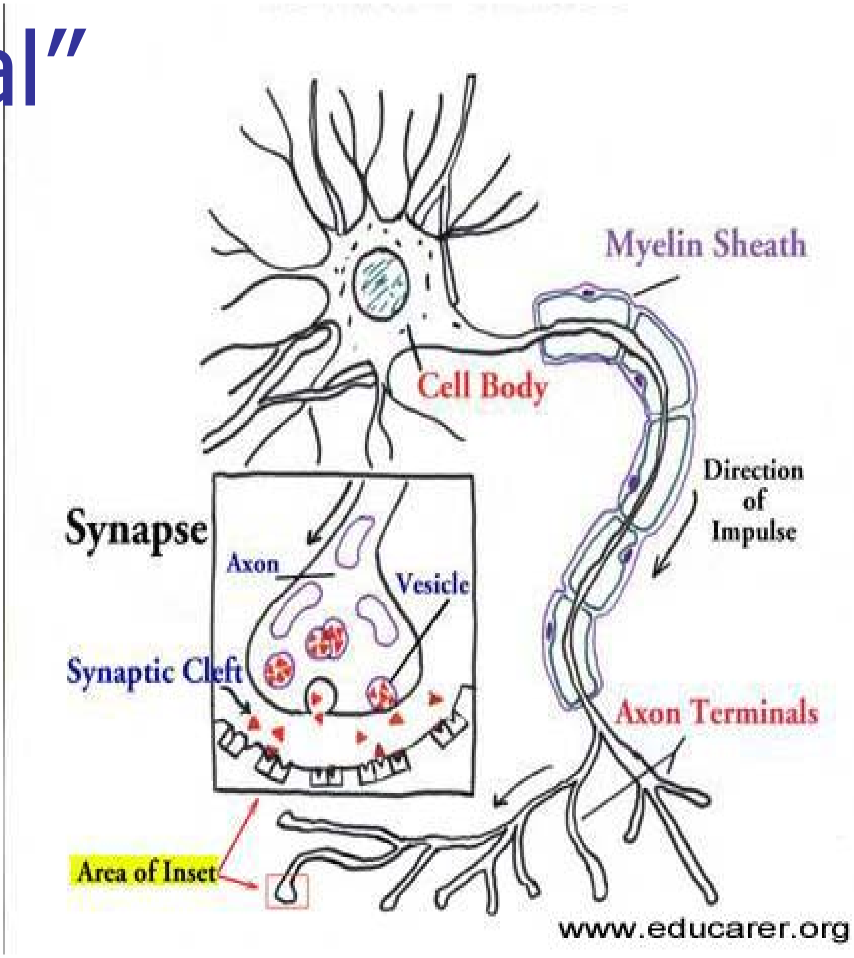
**Disciplines which form the core of AI - inner circle**

**Fields which draw from these disciplines - outer circle.**



# Neuron - "classical"

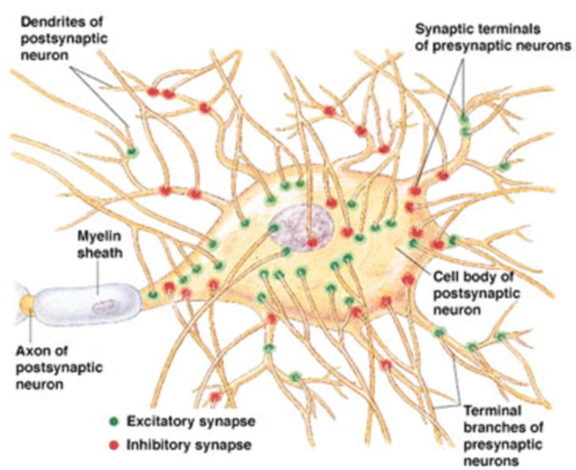
- Dendrites
  - Receiving stations of neurons
  - Don't generate action potentials
- Cell body
  - Site at which information received is integrated
- Axon
  - Generate and relay action potential
  - Terminal
    - Relays information to next neuron in the pathway




http://www.educarer.com/images/brain-nerve-axon.jpg

# Computation in Biological Neuron

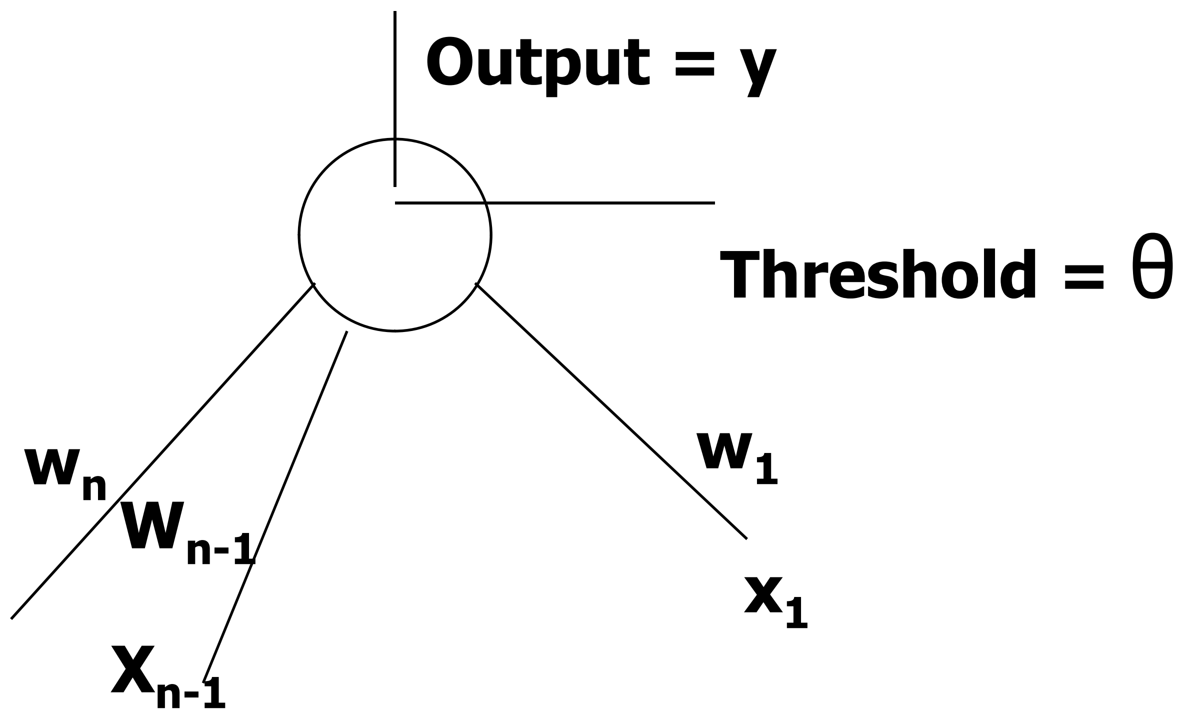
- Incoming signals from synapses are summed up at the soma
- $\Sigma$ , the biological "inner product"
- On crossing a threshold, the cell "fires" generating an action potential in the axon hillock region

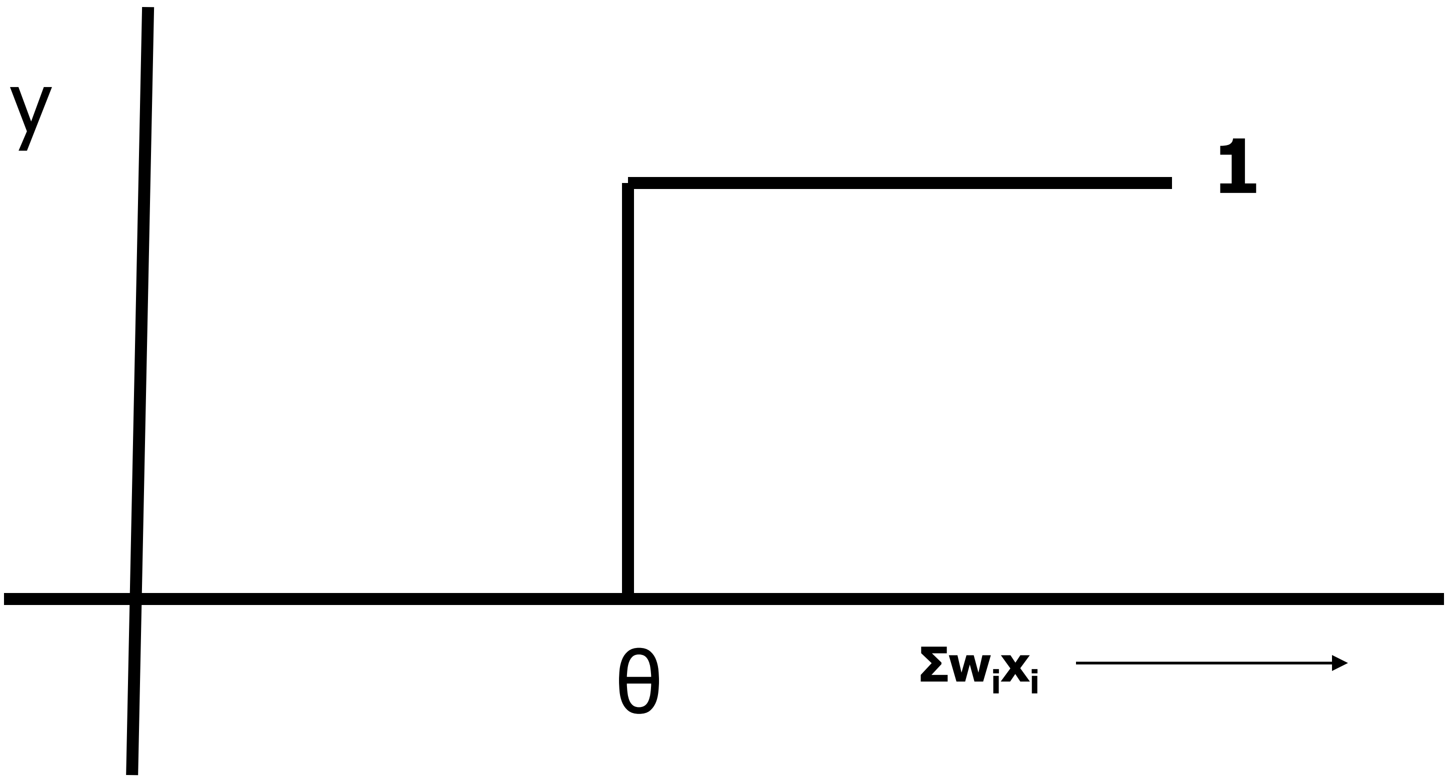



**Synaptic inputs: Artist's conception**

# The Perceptron Model

A perceptron is a computing element with input lines having associated weights and the cell having a threshold value. The perceptron model is motivated by the biological neuron.






**Step function / Threshold function**

**y = 1 for $\Sigma w_i x_i >= \theta$**

**=0 otherwise**

# Features of Perceptron

- Input output behavior is discontinuous and the derivative does not exist at $\mathbf{\Sigma w_i x_i = \theta}$

- $\mathbf{\Sigma w_i x_i - \theta}$ is the net input denoted as net

- Referred to as a linear threshold element - linearity because of **x** appearing with power **1**

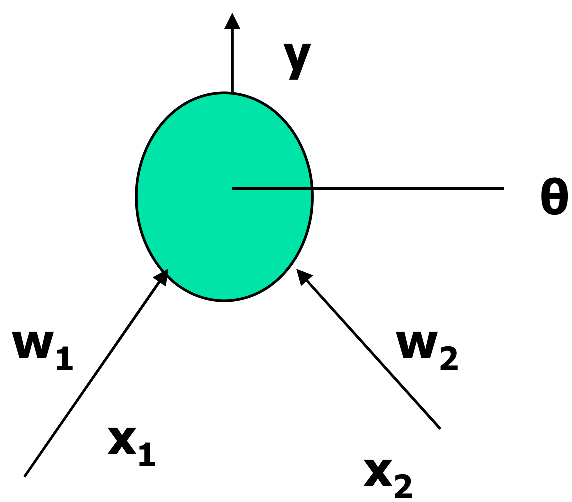
- **y= f(net)**: Relation between y and net is non-linear

# Computation of Boolean functions

**AND of 2 inputs**

| **X1** | **x2** | **y** |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

The parameter values (weights & thresholds) need to be found.



# Computing parameter values

w1 * 0 + w2 * 0 <= θ ➔ θ >= 0; since y=0

w1 * 0 + w2 * 1 <= θ ➔ w2 <= θ; since y=0

w1 * 1 + w2 * 0 <= θ ➔ w1 <= θ; since y=0

w1 * 1 + w2 *1 > θ ➔ w1 + w2 > θ; since y=1

w1 = w2 = = 0.5

satisfy these inequalities and find parameters to be used for computing AND function.

# Other Boolean functions

- **OR can be computed using values of w1 = w2 = 1 and = 0.5**

- **XOR function gives rise to the following inequalities:**

$w1 * 0 + w2 * 0 <= \theta \Rightarrow \theta >= 0$

$w1 * 0 + w2 * 1 > \theta \Rightarrow w2 > \theta$

$w1 * 1 + w2 * 0 > \theta \Rightarrow w1 > \theta$

$w1 * 1 + w2 * 1 <= \theta \Rightarrow w1 + w2 <= \theta$

No set of parameter values satisfy these inequalities.

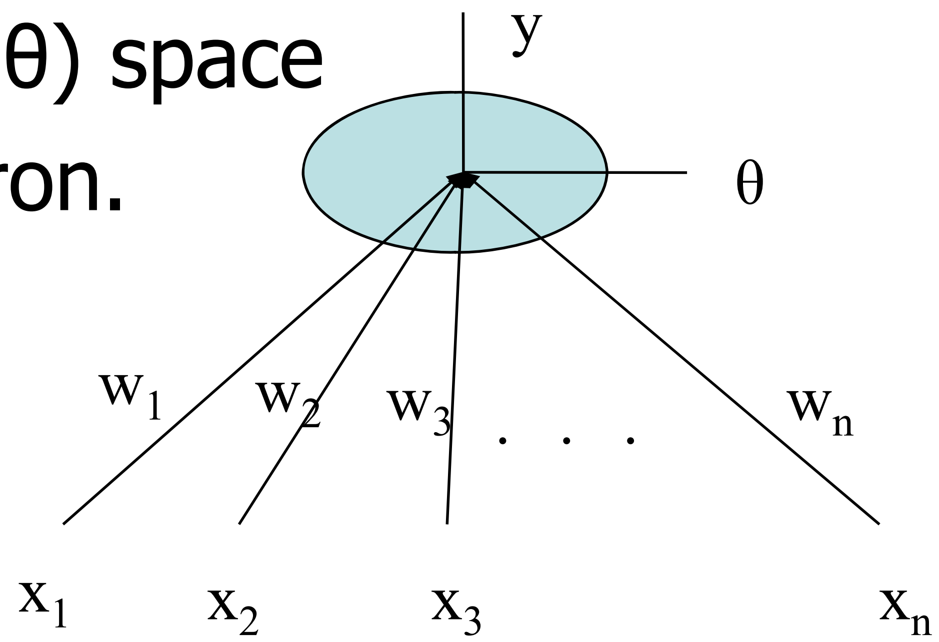
# Threshold functions

| n | # Boolean functions (2^2^n) $(2^{n^2})$ | #Threshold Functions |
|---|---|---|
| 1 | 4 | 4 |
| 2 | 16 | 14 |
| 3 | 256 | 128 |
| 4 | 64K | 1008 |

- Functions computable by perceptrons - threshold functions
- #TF becomes negligibly small for larger values of #BF.
- For n=2, all functions except XOR and XNOR are computable.

# Concept of Hyper-planes

- $\sum w_i x_i = \theta$ defines a linear surface in the $(W,\theta)$ space, where $W=\langle w_1,w_2,w_3,\ldots,w_n\rangle$ is an n-dimensional vector.
- A point in this $(W,\theta)$ space defines a perceptron.



# Perceptron Property

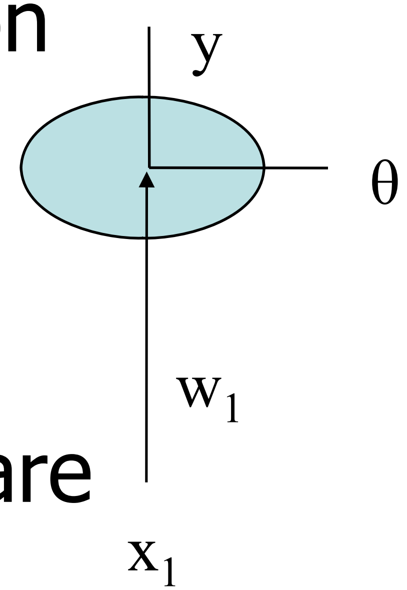
- Two perceptrons may have different parameters but same functional values.

- Example of the simplest perceptron

w.x>0 gives y=1

w.x≤0 gives y=0

Depending on different values of w and θ, four different functions are possible



# Simple perceptron contd.

| x | f1 | f2 | f3 | f4 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 |

**True-Function**

$\theta<0$
$W<0$

**0-function**

$\theta\geq0$
$w\leq0$

**Identity Function**

$\theta\geq0$
$w>0$

**Complement Function**

$\theta<0$
$w\leq0$

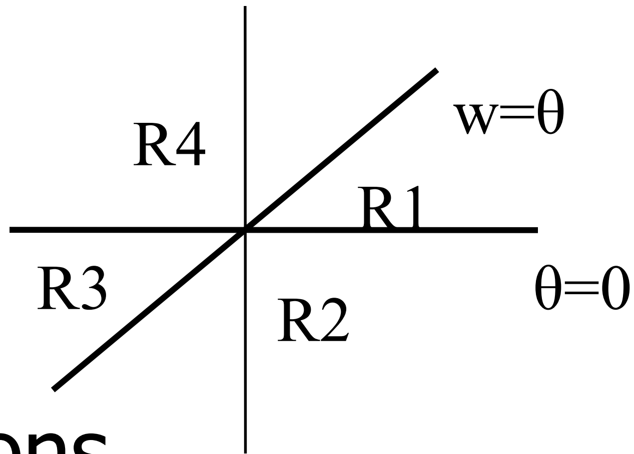
# Counting the number of functions for the simplest perceptron

- For the simplest perceptron, the equation is $w.x=\theta$.

Substituting x=0 and x=1,

we get $\theta=0$ and $w=\theta$.

These two lines intersect to form four regions, which correspond to the four functions.



# Fundamental Observation

- The number of TFs computable by a perceptron is equal to the number of regions produced by $2^n$ hyper-planes,obtained by plugging in the values $<x_1,x_2,x_3,\ldots,x_n>$ in the equation

$$\sum_{i=1}^{n} w_i x_i = \theta$$

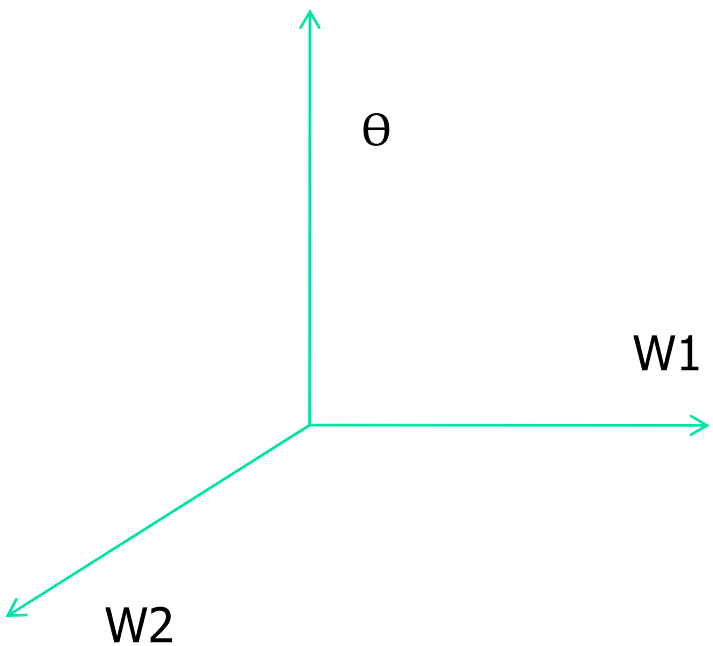
# The geometrical observation

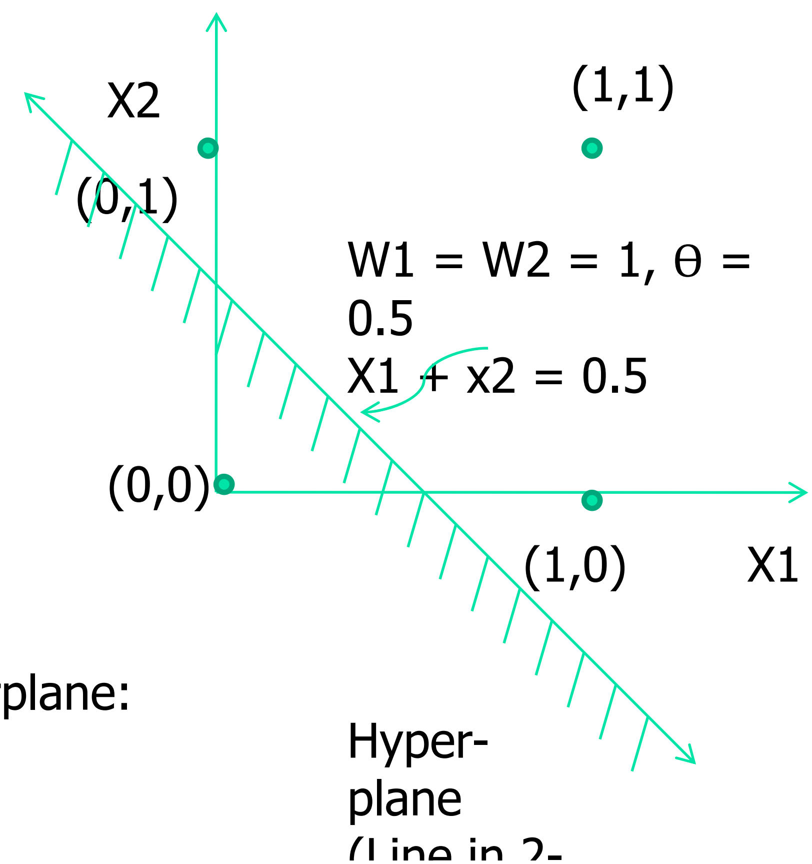
- **Problem:** m linear surfaces called hyper-planes (each hyper-plane is of (d-1)-dim) in d-dim, then what is the max. no. of regions produced by their intersection?

  i.e. $R_{m,d} = ?$

# Co-ordinate Spaces

We work in the $<X_1, X_2>$ space or the $<w_1, w_2, \theta>$ space

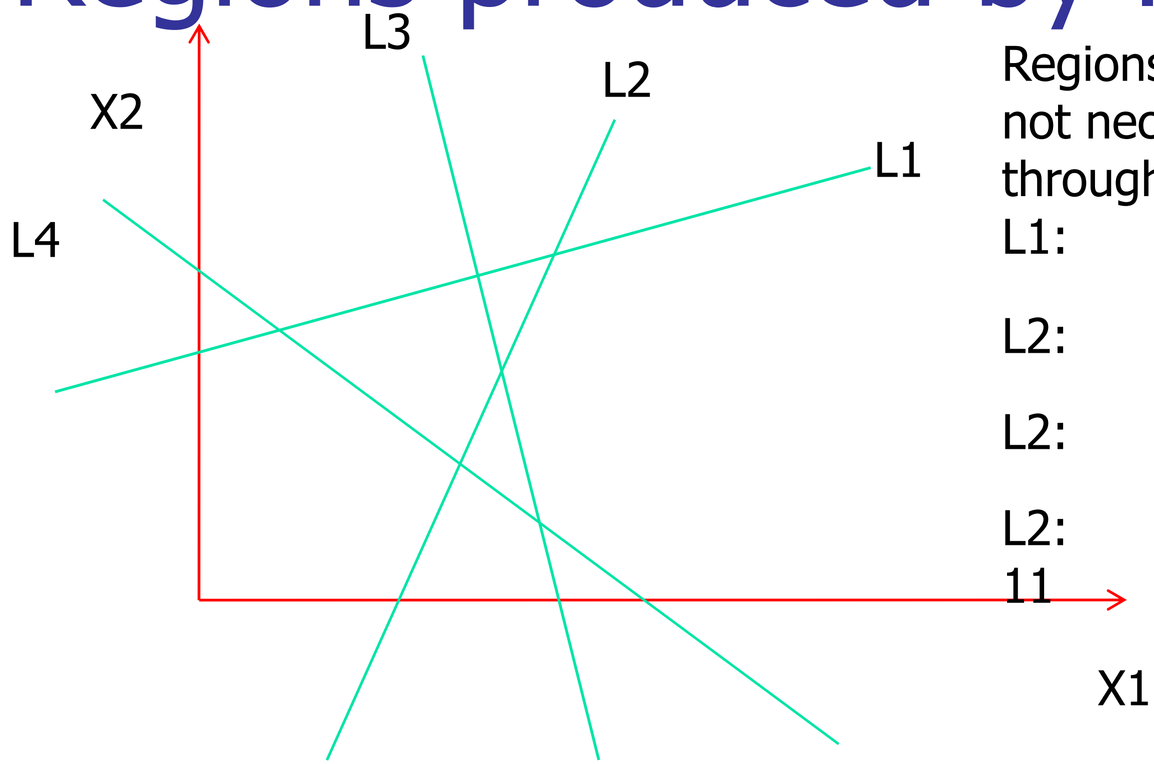



General equation of a Hyperplane:

$\Sigma \, W_i X_i = \theta$

# Regions produced by lines




Regions produced by lines not necessarily passing through origin

L1: 2

L2: 2+2 = 4

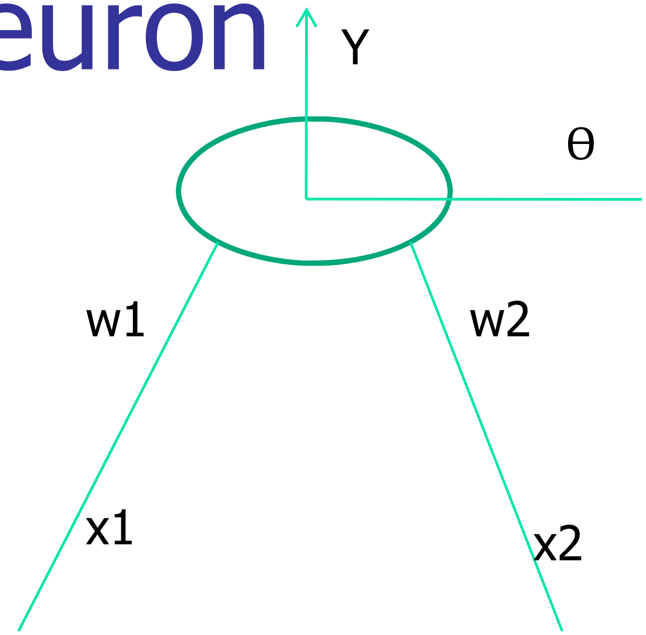
L2: 2+2+3 = 7

L2: 2+2+3+4 = 11

New regions created = Number of intersections on the incoming line by the original lines

Total number of regions = Original number of regions + New regions created

# Number of computable functions by a neuron




$$w1 * x1 + w2 * x2 = \theta$$

$$(0,0) \Rightarrow \theta = 0 : P1$$

$$(0,1) \Rightarrow w2 = \theta : P2$$

$$(1,0) \Rightarrow w1 = \theta : P3$$

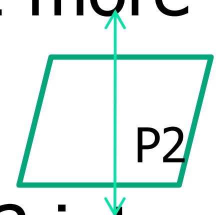
$$(1,1) \Rightarrow w1 + w2 = \theta : P4$$

P1, P2, P3 and P4 are planes in the <W1,W2, θ> space

# Number of computable functions by a neuron (cont…)

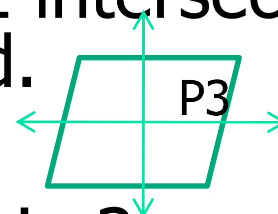
- P1 produces 2 regions
- P2 is intersected by P1 in a line. 2 more new regions are produced.

  Number of regions = 2+2 = 4



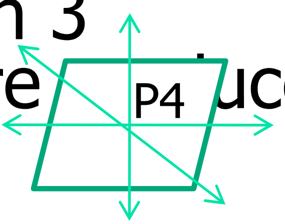

- P3 is intersected by P1 and P2 in 2 intersecting lines. 4 more regions are produced.

  Number of regions = 4 + 4 = 8




- P4 is intersected by P1, P2 and P3 in 3 intersecting lines. 6 more regions are produced.

  Number of regions = 8 + 6 = 14




- Thus, a single neuron can compute 14 Boolean functions which are linearly separable.

# Points in the same region

If

$W_1 * X_1 + W_2 * X_2 > \theta$

$W_1' * X_1 + W_2' * X_2 > \theta'$

Then

If $<W_1, W_2, \theta>$ and $<W_1', W_2', \theta'>$ share a region then they compute the same function