CS344: Introduction to Artificial Intelligence (associated lab: CS386)

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Lecture 24: Perceptrons and their computing power (cntd)

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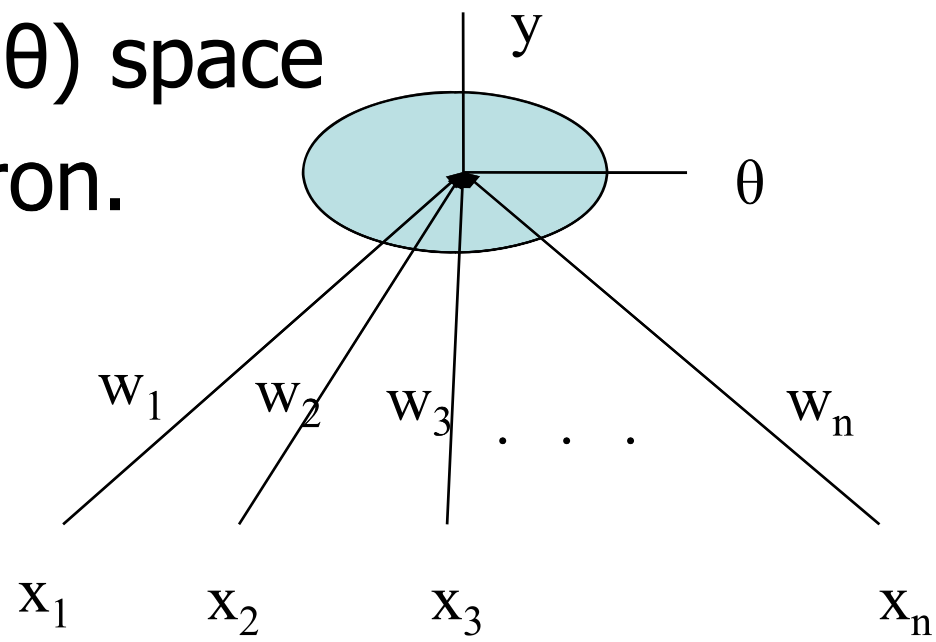
Threshold functions

n	# Boolean functions (2^2^n)	#Threshold Functions (2^{n^2})
1	4	4
2	16	14
3	256	128
4	64K	1008

- **Functions computable by perceptrons - threshold functions**
- **#TF becomes negligibly small for larger values of #BF.**
- **For n=2, all functions except XOR and XNOR are computable.**

Concept of Hyper-planes

- $\sum w_i x_i = \theta$ defines a linear surface in the (W,θ) space, where $W=\langle w_1,w_2,w_3,\ldots,w_n\rangle$ is an n-dimensional vector.
- A point in this (W,θ) space defines a perceptron.



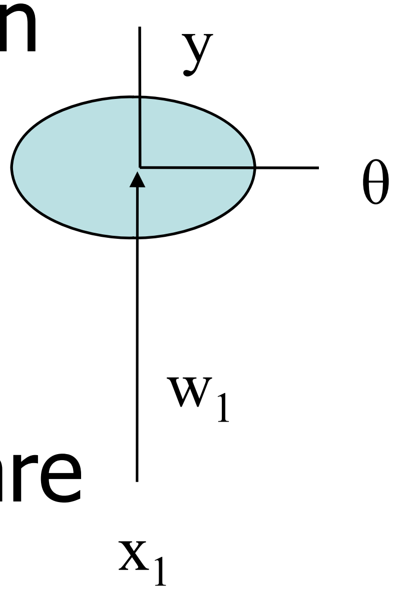
Perceptron Property

- Two perceptrons may have different parameters but same function
- Example of the simplest perceptron

$$w.x>0 \text{ gives } y=1$$

$$w.x\leq 0 \text{ gives } y=0$$

Depending on different values of w and θ, four different functions are possible



Simple perceptron contd.

x	f1	f2	f3	f4
0	0	0	1	1
1	0	1	0	1

<u>True-Function</u>

$\theta<0$
$W<0$

<u>0-function</u>

$\theta\geq 0$
$w\leq 0$

<u>Identity Function</u>

$\theta\geq 0$
$w>0$

<u>Complement Function</u>

$\theta<0$
$w\leq 0$

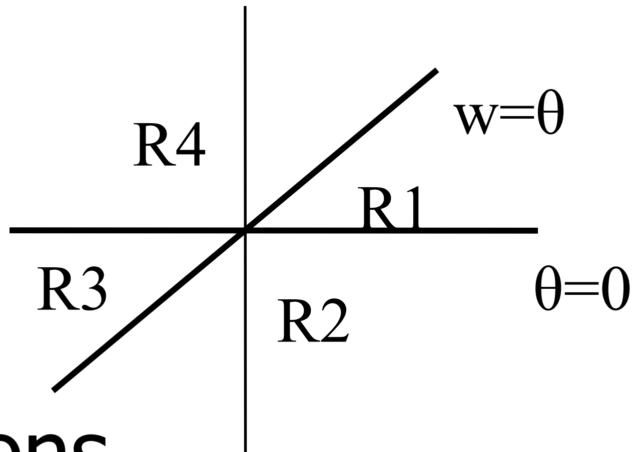
Counting the number of functions for the simplest perceptron

- For the simplest perceptron, the equation is $w.x=\theta$.

Substituting x=0 and x=1,

we get $\theta=0$ and $w=\theta$.

These two lines intersect to form four regions, which correspond to the four functions.



Fundamental Observation

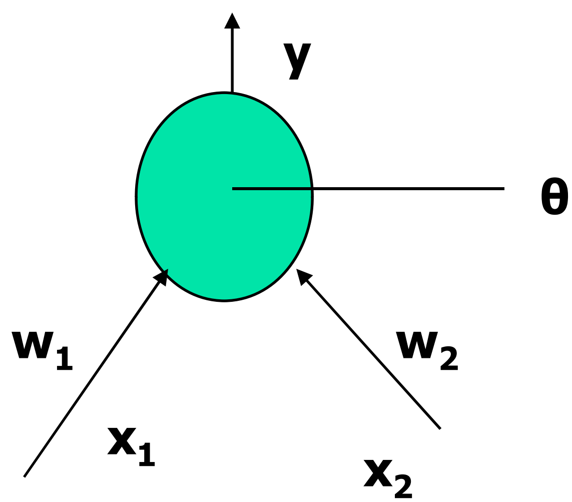
- The number of TFs computable by a perceptron is equal to the number of regions produced by 2^n hyper-planes,obtained by plugging in the values $<x_1,x_2,x_3,\ldots,x_n>$ in the equation

$$\sum_{i=1}^{n} w_i x_i = \theta$$

AND of 2 inputs

X1	x2	y
0	0	0
0	1	0
1	0	0
1	1	1

The parameter values (weights & thresholds) need to be found.



Constraints on w1, w2 and θ

$w1 * 0 + w2 * 0 <= \theta \Rightarrow \theta >= 0$; since $y=0$

$w1 * 0 + w2 * 1 <= \theta \Rightarrow w2 <= \theta$; since $y=0$

$w1 * 1 + w2 * 0 <= \theta \Rightarrow w1 <= \theta$; since $y=0$

$w1 * 1 + w2 * 1 > \theta \Rightarrow w1 + w2 > \theta$; since $y=1$

$w1 = w2 = \quad = 0.5$

These inequalities are satisfied by ONE particular region

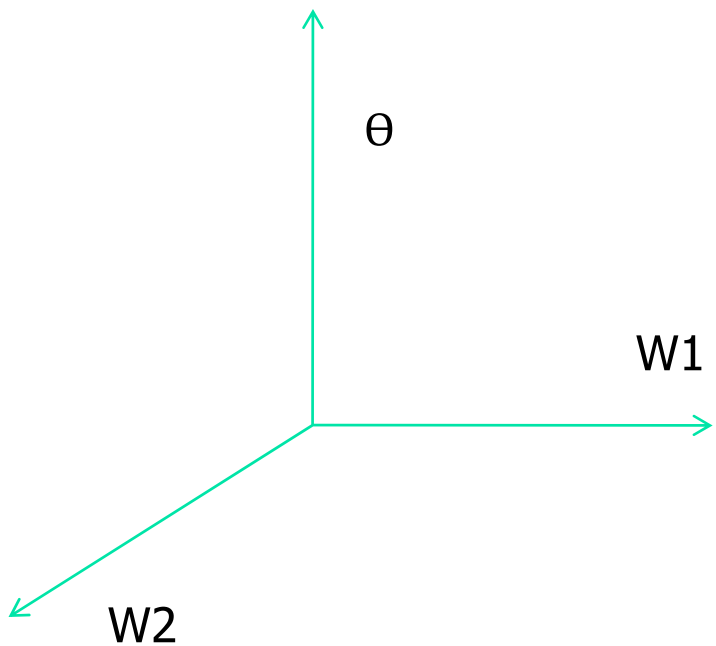
The geometrical observation

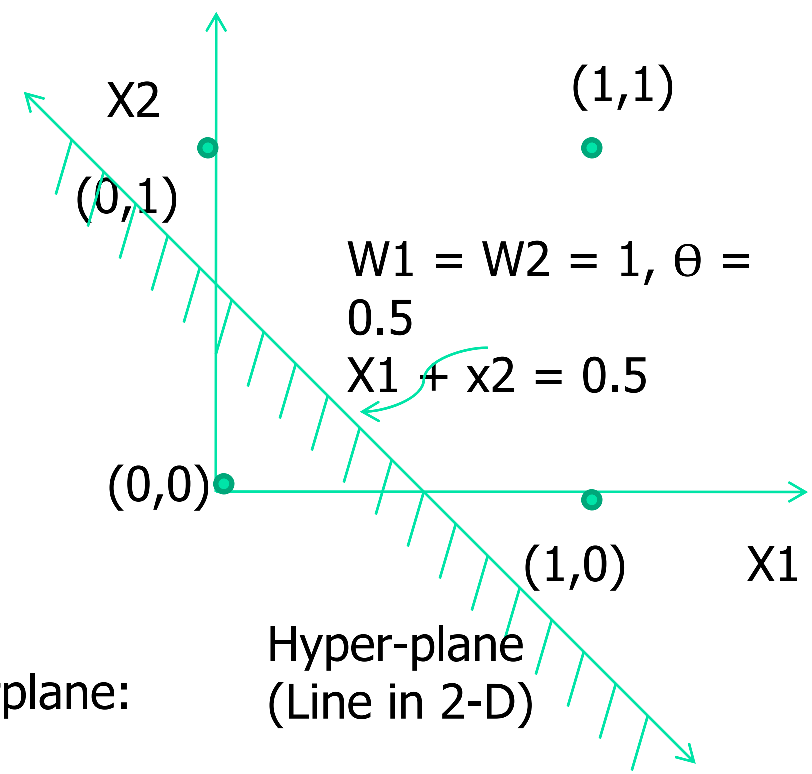
- **Problem:** *m* linear surfaces called hyper-planes (each hyper-plane is of *(d-1)*-dim) in d-dim, then what is the max. no. of regions produced by their intersection?

  i.e., $R_{m,d} = ?$

Co-ordinate Spaces

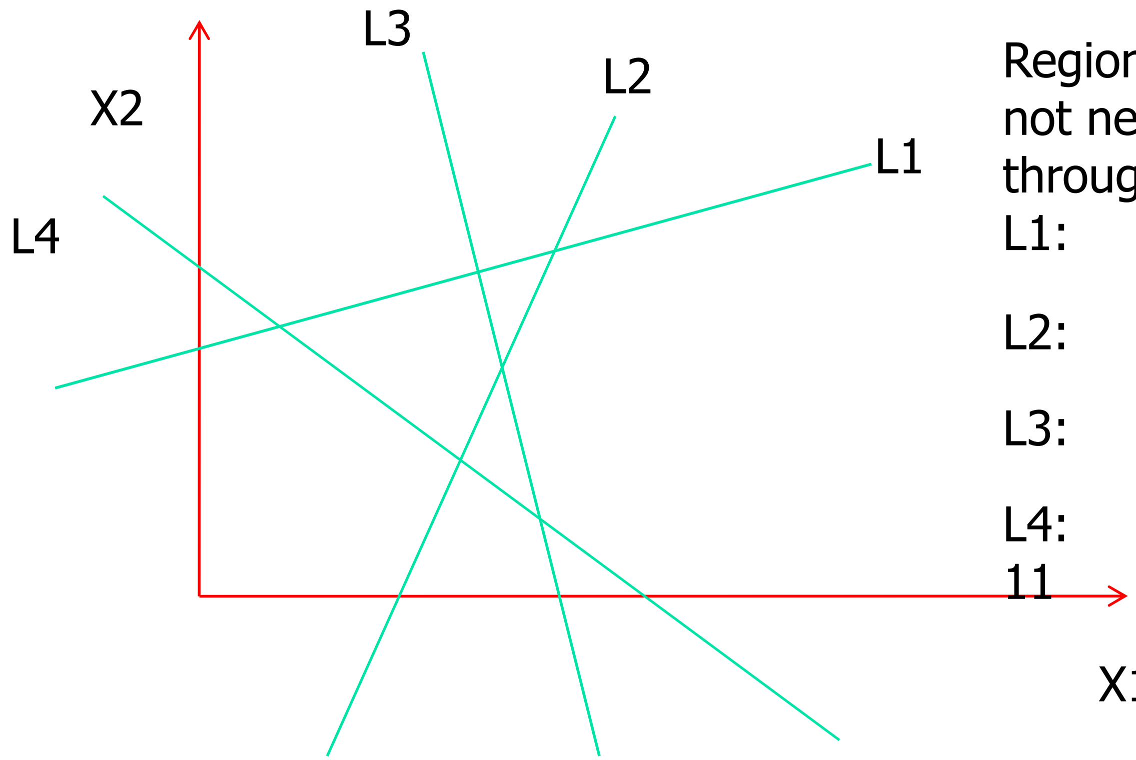
We work in the $\langle X_1, X_2 \rangle$ space or the $\langle w_1, w_2, \theta \rangle$ space





General equation of a Hyperplane:
$\Sigma W_i X_i = \theta$

Regions produced by lines



Regions produced by lines not necessarily passing through origin

L1: 2

L2: 2+2 = 4

L3: 2+2+3 = 7

L4: 2+2+3+4 = 11

New regions created = Number of intersections on the incoming line by the original lines

Total number of regions = Original number of regions + New regions created

Number of computable functions by a neuron

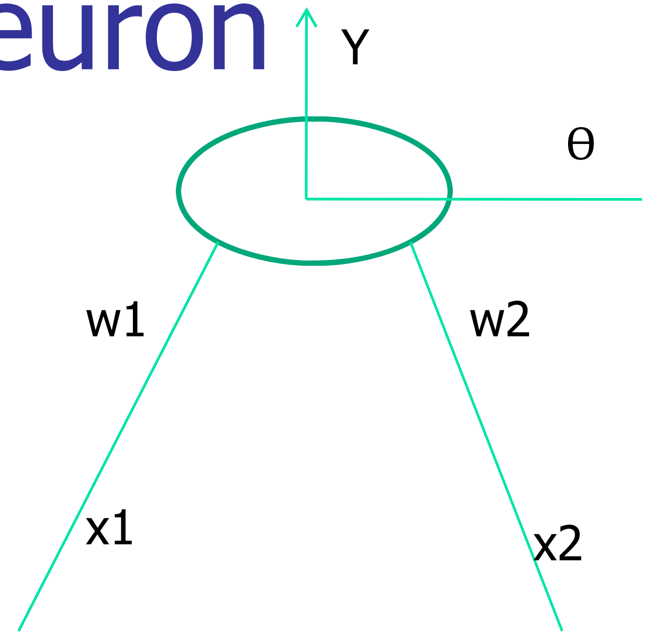
$$w1 * x1 + w2 * x2 = \theta$$

$$(0,0) \Rightarrow \theta = 0 : P1$$

$$(0,1) \Rightarrow w2 = \theta : P2$$

$$(1,0) \Rightarrow w1 = \theta : P3$$

$$(1,1) \Rightarrow w1 + w2 = \theta : P4$$

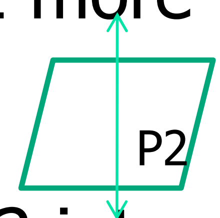


P1, P2, P3 and P4 are planes in the <W1,W2, θ> space

Number of computable functions by a neuron (cont…)

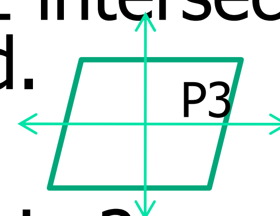
- P1 produces 2 regions
- P2 is intersected by P1 in a line. 2 more new regions are produced.

  Number of regions = 2+2 = 4



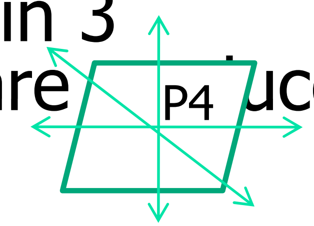
- P3 is intersected by P1 and P2 in 2 intersecting lines. 4 more regions are produced.

  Number of regions = 4 + 4 = 8



- P4 is intersected by P1, P2 and P3 in 3 intersecting lines. 6 more regions are produced.

  Number of regions = 8 + 6 = 14



- Thus, a single neuron can compute 14 Boolean functions which are linearly separable.

Points in the same region

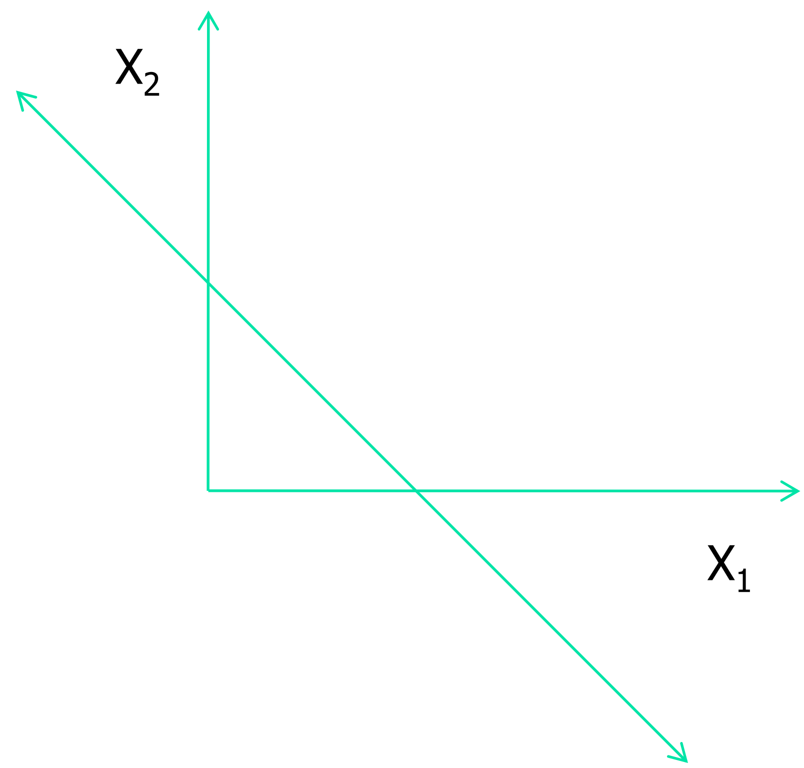
If

$W_1 * X_1 + W_2 * X_2 > \theta$

$W_1' * X_1 + W_2' * X_2 > \theta'$

Then

If $<W_1, W_2, \theta>$ and $<W_1', W_2', \theta'>$ share a region then they compute the same function



No. of Regions produced by Hyperplanes

Number of regions founded by n hyperplanes in d-dim passing through origin is given by the following recurrence relation

$$R_{n,d} = R_{n-1,d} + R_{n-1,d-1}$$

we use generating function as an operating function

Boundary condition:

$R_{1,d} = 2$ 1 hyperplane in d-dim

$R_{n,1} = 2$ n hyperplanes in 1-dim, Reduce to n points thru origin

The generating function is

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} R_{n,d} \cdot x^n y^d$$

From the recurrence relation we have,

$$R_{n,d} - R_{n-1,d} - R_{n-1,d-1} = 0$$

$R_{n-1,d}$ corresponds to 'shifting' n by 1 place, => multiplication by x

$R_{n-1,d-1}$ corresponds to 'shifting' n and d by 1 place => multiplication by xy

On expanding $f(x,y)$ we get

$$\begin{aligned} f(x,y) = {} & R_{1,1} \cdot xy + R_{1,2} \cdot x y^2 + R_{1,3} \cdot x y^3 + \ldots + R_{1,d} \cdot x y^d + \ldots..\infty \\ & + R_{2,1} \cdot x^2 y + R_{2,2} \cdot x^2 y^2 + R_{2,3} \cdot x^2 y^3 + \ldots + R_{2,d} \cdot x^2 y^d + \ldots..\infty \\ & \ldots.. \\ & + R_{n,1} \cdot x^n y + R_{n,2} \cdot x^n y^2 + R_{n,3} \cdot x^n y^3 + \ldots + R_{n,d} \cdot x^n y^d + \ldots..\infty \end{aligned}$$

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} R_{n,d} \cdot x^n y^d$$

$$x \cdot f(x,y) = \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} R_{n,d} \cdot x^{n+1} y^d = \sum_{n=2}^{\infty}\sum_{d=1}^{\infty} R_{n-1,d} \cdot x^n y^d$$

$$xy \cdot f(x,y) = \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} R_{n,d} \cdot x^{n+1} y^{d+1} = \sum_{n=2}^{\infty}\sum_{d=2}^{\infty} R_{n-1,d-1} \cdot x^n y^d$$

$$x \cdot f(x,y) = \sum_{n=2}^{\infty}\sum_{d=2}^{\infty} R_{n-1,d} \cdot x^n y^d + \sum_{n=2}^{\infty} R_{n-1,1} \cdot x^n y$$

$$= \sum_{n=2}^{\infty}\sum_{d=2}^{\infty} R_{n-1,d} \cdot x^n y^d + 2 \cdot \sum_{n=2}^{\infty} x^n y$$

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{d=1}^{\infty}R_{n,d}\cdot x^n y^d$$

$$=\sum_{n=2}^{\infty}\sum_{d=2}^{\infty}R_{n,d}\cdot x^n y^d+\sum_{d=1}^{\infty}R_{1,d}\cdot xy^d+\sum_{n=1}^{\infty}R_{n,1}\cdot x^n y-R_{1,1}\cdot xy$$

$$=\sum_{n=2}^{\infty}\sum_{d=2}^{\infty}R_{n,d}\cdot x^n y^d+2x\cdot\sum_{d=1}^{\infty}xy^d+2y\cdot\sum_{n=1}^{\infty}x^n y-2xy$$

After all this expansion,

$$f(x,y)-x\cdot f(x,y)-xy\cdot f(x,y)$$

$$=\sum_{n=2}^{\infty}\sum_{d=2}^{\infty}(R_{n,d}-R_{n-1,d}-R_{n-1,d-1})x^n y^d$$

$$+2y\cdot\sum_{n=1}^{\infty}x^d-2xy-2y\cdot\sum_{n=2}^{\infty}x+2x\cdot\sum_{d=1}^{\infty}y^d$$

$$=2x\cdot\sum_{d=1}^{\infty}y^d$$

since other two terms become zero

This implies

$$[1-x-xy]\,f(x,y) = 2x\cdot\sum_{d=1}^{\infty} y^d$$

$$f(x,y) = \frac{1}{[1-x(1-y)]}\cdot 2x\cdot\sum_{d=1}^{\infty} y^d$$

$$= 2x\cdot[y+y^2+y^3+\ldots+y^d+\ldots..\infty]\cdot$$

$$[1+x(1+y)+x^2(1+y)^2+\ldots+x^d(1+y)^d+\ldots..\infty]$$

also we have,

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{d=1}^{\infty} R_{n,\,d}\cdot x^n y^d$$

Comparing coefficients of each term in RHS we get, Comparing co-efficients we get

$$R_{n,\,d} = \sum_{i=0}^{d-1} C_i^{n-1}$$