CS344: Introduction to Artificial Intelligence
(associated lab: CS386)

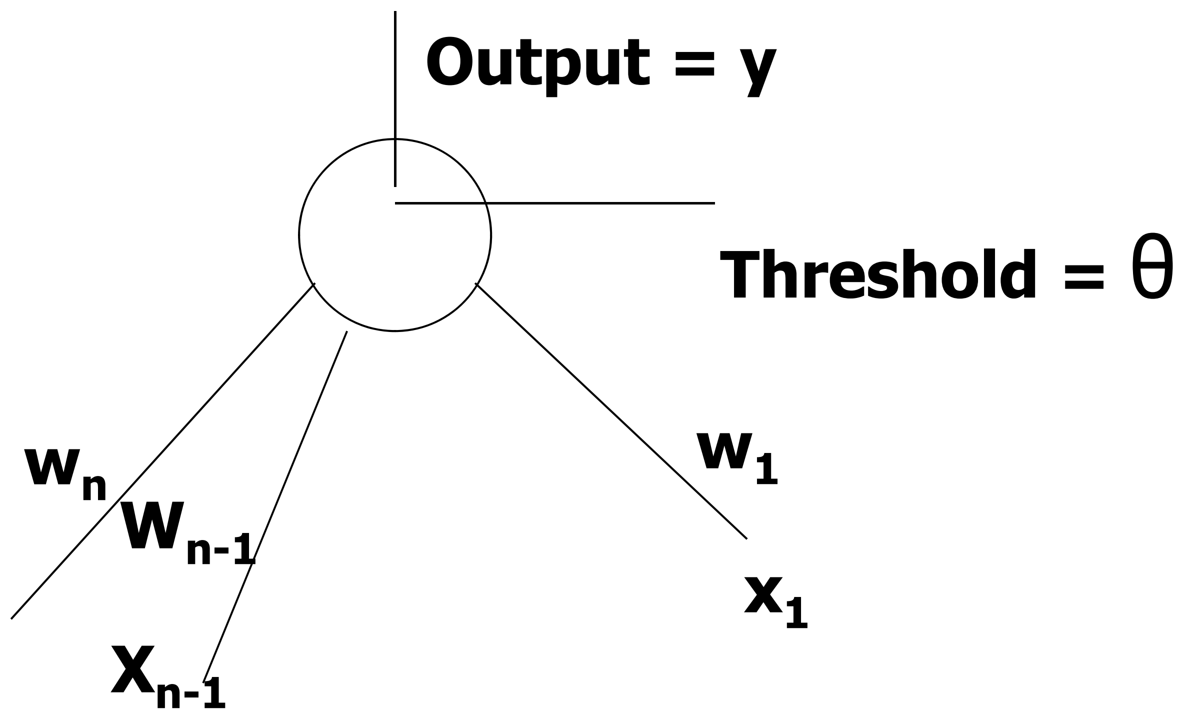
Pushpak Bhattacharyya
CSE Dept.,
IIT Bombay

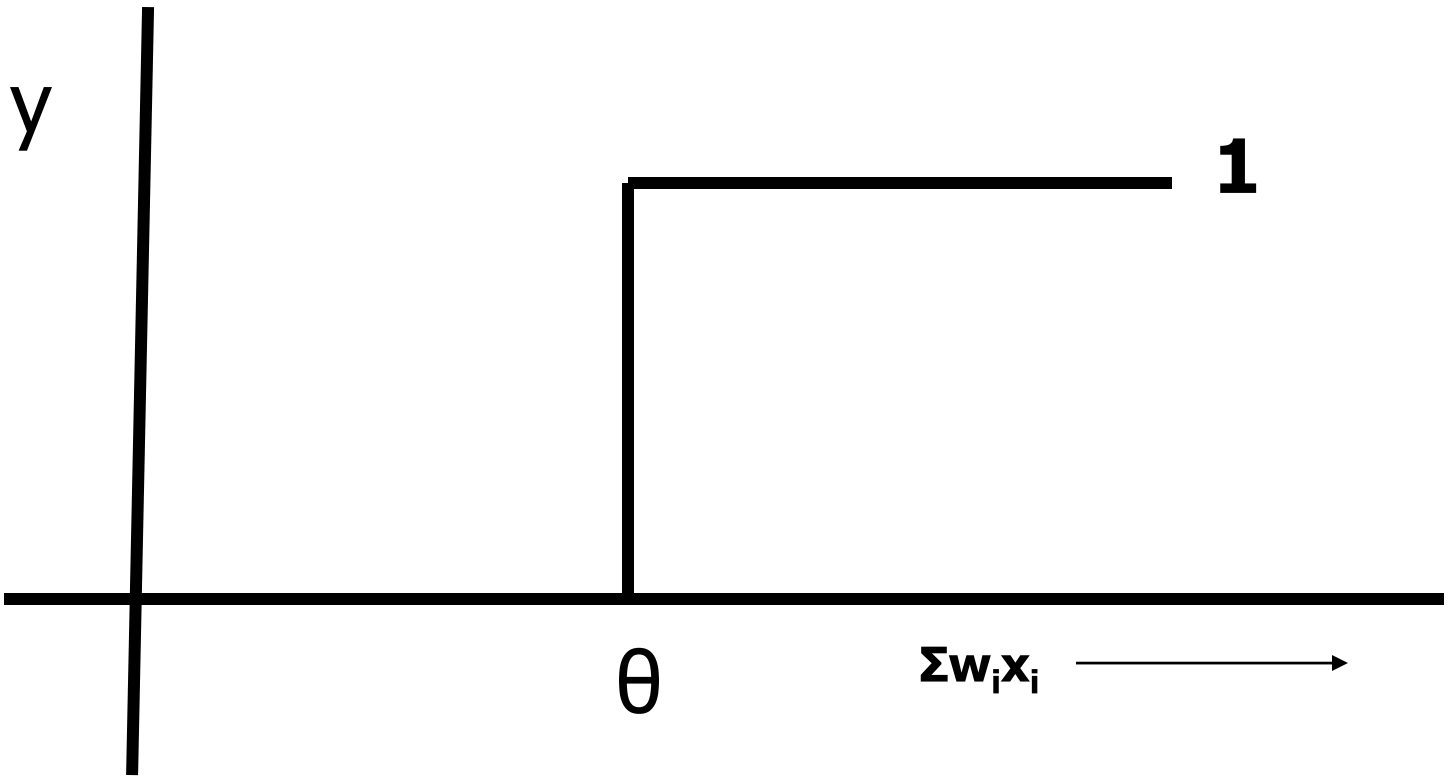
Lecture 29: Perceptron training and convergence

22nd March, 2011

The Perceptron Model

A perceptron is a computing element with input lines having associated weights and the cell having a threshold value. The perceptron model is motivated by the biological neuron.





Step function / Threshold function

y = 1 for $\Sigma w_i x_i >= \theta$

=0 otherwise

Features of Perceptron

- Input output behavior is discontinuous and the derivative does not exist at $\mathbf{\Sigma w_i x_i = \theta}$

- $\mathbf{\Sigma w_i x_i - \theta}$ is the net input denoted as net

- Referred to as a linear threshold element - linearity because of **x** appearing with power **1**

- **y= f(net)**: Relation between y and net is non-linear

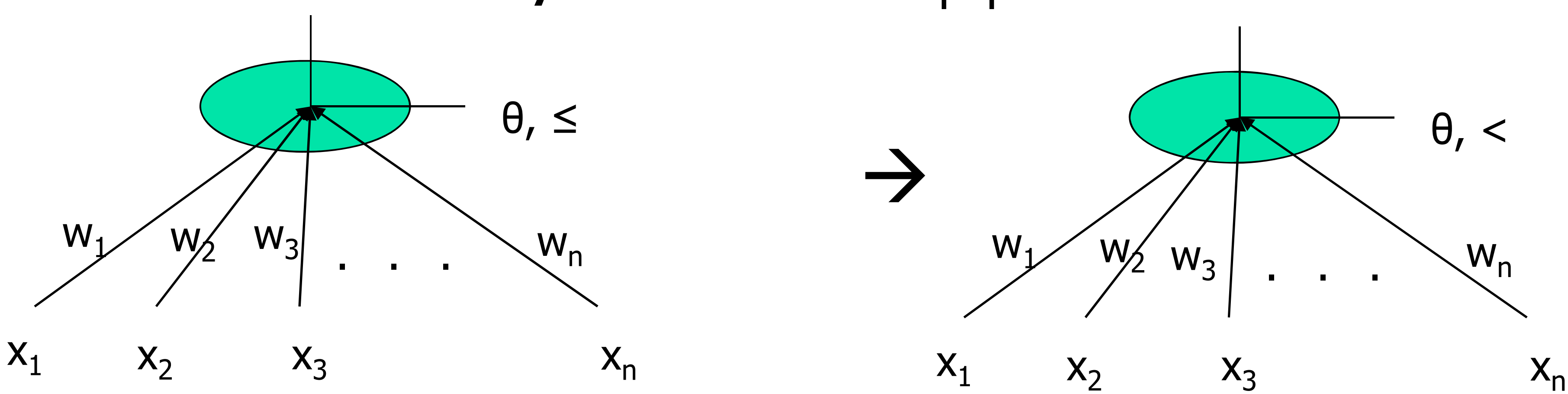
Perceptron Training Algorithm (PTA)

Preprocessing:

1. The computation law is modified to

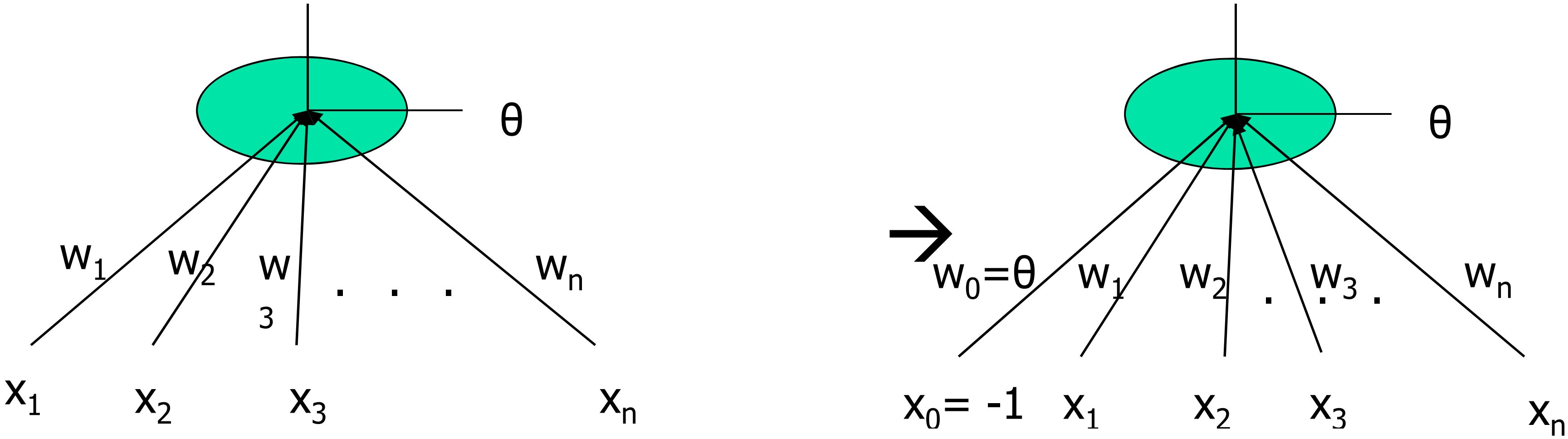
$$y = 1 \text{ if } \sum w_i x_i > \theta$$

$$y = o \text{ if } \sum w_i x_i < \theta$$



PTA – preprocessing cont…

2. Absorb θ as a weight



3. Negate all the zero-class examples

Example to demonstrate preprocessing

- **OR perceptron**

1-class <1,1> , <1,0> , <0,1>

0-class <0,0>

Augmented x vectors:-

1-class <-1,1,1> , <-1,1,0> , <-1,0,1>

0-class <-1,0,0>

Negate 0-class:- <1,0,0>

Example to demonstrate preprocessing cont..

Now the vectors are

	x_0	x_1	x_2
X_1	-1	0	1
X_2	-1	1	0
X_3	-1	1	1
X_4	1	0	0

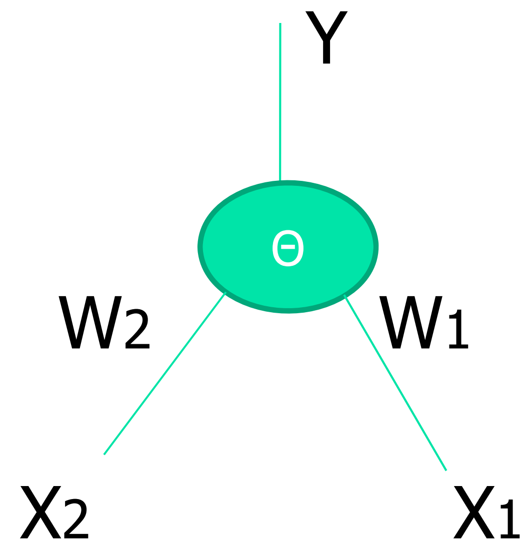
Perceptron Training Algorithm

1. Start with a random value of w
   ex: <0,0,0…>
2. Test for $wx_i > 0$
   If the test succeeds for i=1,2,…n
   then return w
3. Modify w, $w_{next} = w_{prev} + x_{fail}$

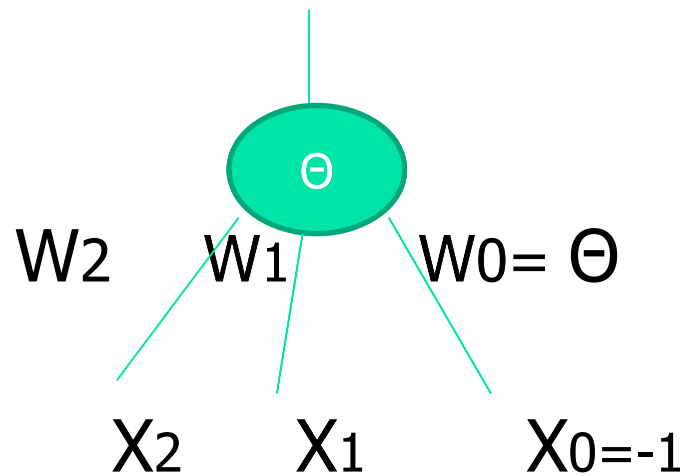
PTA on NAND

NAND:

X_2	X_1	Y
0	0	1
0	1	1
1	0	1
1	1	1



Converted To



Preprocessing

NAND Augmented:

X_2	X_1	X_0	Y
0	0	-1	1
0	1	-1	1
1	0	-1	1
1	1	-1	0

NAND-0 class Negated

	X_2	X_1	X_0
V_0:	0	0	-1
V_1:	0	1	-1
V_2:	1	0	-1
V_3:	-1	1	-1

Vectors for which W=<W2 W1 W0> has to be found such that W. Vi > 0

PTA Algo steps

Algorithm:

1. Initialize and Keep adding the failed vectors until W. Vi > 0 is true.

Step 0: W = <0, 0, 0>

W_1 = <0, 0, 0> + <0, 0, -1> {V_0 Fails}

= <0, 0, -1>

W_2 = <0, 0, -1> + <-1, -1, 1> {V_3 Fails}

= <-1, -1, 0>

W_3 = <-1, -1, 0> + <0, 0, -1> {V_0 Fails}

= <-1, -1, -1>

W_4 = <-1, -1, -1> + <0, 1, -1> {V_1 Fails}

= <-1, 0, -2>

Trying convergence

W_5 = <-1, 0, -2> + <-1, -1, -1> {V_3 Fails}

= <-2, -1, -1>

W_6 = <-2, -1, -1> + <0, 1, -1> {V_1 Fails}

= <-2, 0, -2>

W_7 = <-2, 0, -2> + <1, 0, -1> {V_0 Fails}

= <-1, 0, -3>

W_8 = <-1, 0, -3> + <-1, -1, -1> {V_3 Fails}

= <-2, -1, -2>

W_9 = <-2, -1, -2> + <1, 0, -1> {V_2 Fails}

= <-1, -1, -3>

Trying convergence

$W_{10} = <-1, -1, -3> + <-1, -1, -1>$ $\{V_3 \text{ Fails}\}$

$= <-2, -2, -2>$

$W_{11} = <-2, -2, -2> + <0, 1, -1>$ $\{V_1 \text{ Fails}\}$

$= <-2, -1, -3>$

$W_{12} = <-2, -1, -3> + <-1, -1, -1>$ $\{V_3 \text{ Fails}\}$

$= <-3, -2, -2>$

$W_{13} = <-3, -2, -2> + <0, 1, -1>$ $\{V_1 \text{ Fails}\}$

$= <-3, -1, -3>$

$W_{14} = <-3, -1, -3> + <0, 1, -1>$ $\{V_2 \text{ Fails}\}$

$= <-2, -1, -4>$

Converged!

$W_{15} = <-2, -1, -4> + <-1, -1, -1>$ {V_3 Fails}

$= <-3, -2, -3>$

$W_{16} = <-3, -2, -3> + <1, 0, -1>$ {V_2 Fails}

$= <-2, -2, -4>$

$W_{17} = <-2, -2, -4> + <-1, -1, -1>$ {V_3 Fails}

$= <-3, -3, -3>$

$W_{18} = <-3, -3, -3> + <0, 1, -1>$ {V_1 Fails}

$= <-3, -2, -4>$

$W_2 = -3, \; W_1 = -2, \; W_0 = \Theta = -4$

PTA convergence

Statement of Convergence of PTA

- Statement:

  Whatever be the initial choice of weights and whatever be the vector chosen for testing, PTA converges if the vectors are from a linearly separable function.

Proof of Convergence of PTA

- Suppose w_n is the weight vector at the n^{th} step of the algorithm.
- At the beginning, the weight vector is w_0
- Go from w_i to w_{i+1} when a vector X_j fails the test $w_i X_j > 0$ and update w_i as

  $$w_{i+1} = w_i + X_j$$

- Since Xjs form a linearly separable function,

  $$\exists\ w^* \text{ s.t. } w^* X_j > 0\ \forall j$$

Proof of Convergence of PTA (cntd.)

- Consider the expression

$$G(w_n) = \frac{w_n \cdot w^*}{|w_n|}$$

where w_n = weight at nth iteration

- $G(w_n) = \dfrac{|w_n| \cdot |w^*| \cdot \cos\theta}{|w_n|}$

where θ = angle between w_n and w^*

- $G(w_n) = |w^*| \cdot \cos\theta$
- $G(w_n) \leq |w^*|$ (as $-1 \leq \cos\theta \leq 1$)

Behavior of Numerator of G

$w_n \,.\, w^* = (w_{n-1} + X^{n-1}_{fail}) \,.\, w^*$

= $w_{n-1} \,.\, w^* + X^{n-1}_{fail} \,.\, w^*$

= $(w_{n-2} + X^{n-2}_{fail}) \,.\, w^* + X^{n-1}_{fail} \,.\, w^* \,.....$

= $w_0 \,.\, w^* + (X^0_{fail} + X^1_{fail} + ... + X^{n-1}_{fail}). \, w^*$

$w^*.X^i_{fail}$ is always positive: note carefully

- Suppose $|X_j| \geq \delta$, where δ is the minimum magnitude.
- Num of G $\geq |w_0 \,.\, w^*| + n\, \delta \,.\, |w^*|$
- So, numerator of G grows with n.

Behavior of Denominator of G

- $|w_n| = \sqrt{w_n \, . \, w_n}$
- $= \sqrt{(w_{n-1} + X^{n-1}_{fail})^2}$
- $= \sqrt{(w_{n-1})^2 + 2.\, w_{n-1}.\, X^{n-1}_{fail} + (X^{n-1}_{fail})^2}$
- $\leq \sqrt{(w_{n-1})^2 + (X^{n-1}_{fail})^2}$ (as $w_{n-1}.\, X^{n-1}_{fail} \leq 0$)
- $\leq \sqrt{(w_0)^2 + (X^0_{fail})^2 + (X^1_{fail})^2 + ... + (X^{n-1}_{fail})^2}$

- $|X_j| \leq \rho$ (max magnitude)
- So, Denom $\leq \sqrt{(w_0)^2 + n\rho^2}$

Some Observations

- Numerator of G grows as n
- Denominator of G grows as $\sqrt{n}$

  => Numerator grows faster than denominator
- If PTA does not terminate, $G(w_n)$ values will become unbounded.

Some Observations contd.

- But, as $|G(w_n)| \leq |w^*|$ which is finite, this is impossible!
- Hence, PTA has to converge.
- Proof is due to Marvin Minsky.

Convergence of PTA proved

- *Whatever be the initial choice of weights and whatever be the vector chosen for testing, PTA converges if the vectors are from a linearly separable function.*