CS344: Introduction to Artificial Intelligence (associated lab: CS386)

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Lecture 3: A* and its properties

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Search building blocks

- State Space : Graph of states (Express constraints and parameters of the problem)
- Operators : Transformations applied to the states.
- Start state : S_0 (Search starts from here)
- Goal state : $\{G\}$ - Search terminates here.
- Cost : Effort involved in using an operator.
- Optimal path : Least cost path

Examples

Problem 1 : 8 – puzzle

4	3	6
2	1	8
7		5

S

1	2	3
4	5	6
7	8	

G

Tile movement represented as the movement of the blank space.

Operators:

L : Blank moves left

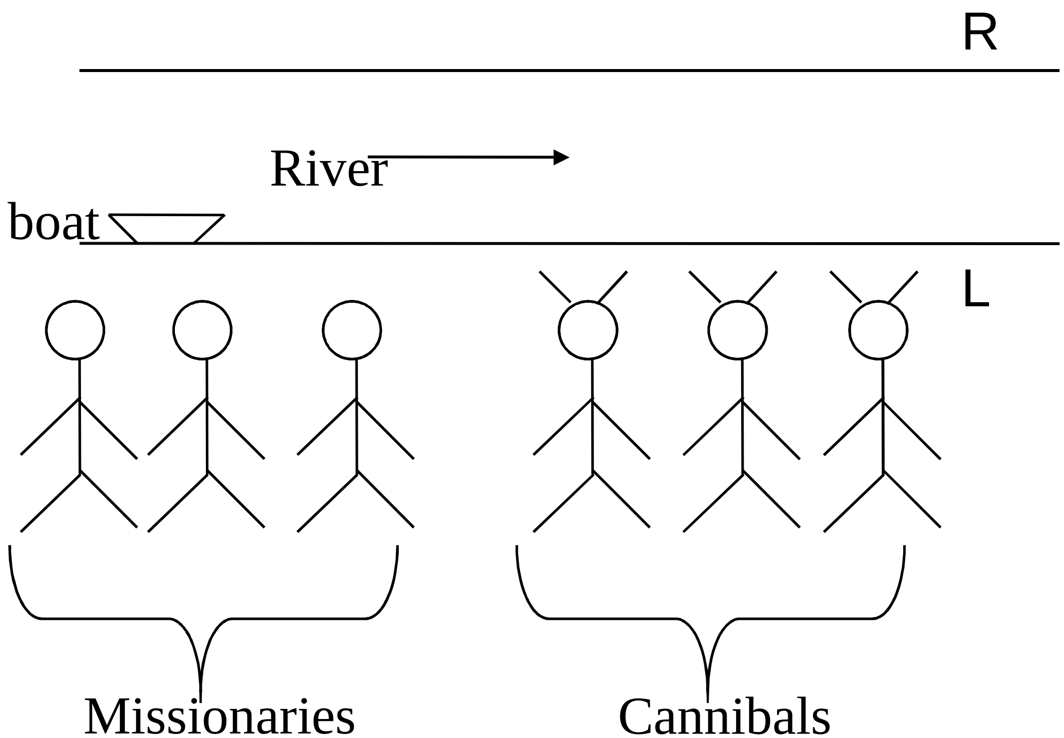
R : Blank moves right

U : Blank moves up

D : Blank moves down

$$C(L) = C(R) = C(U) = C(D) = 1$$

Problem 2: Missionaries and Cannibals



<u>Constraints</u>

- The boat can carry at most 2 people
- On no bank should the cannibals outnumber the missionaries

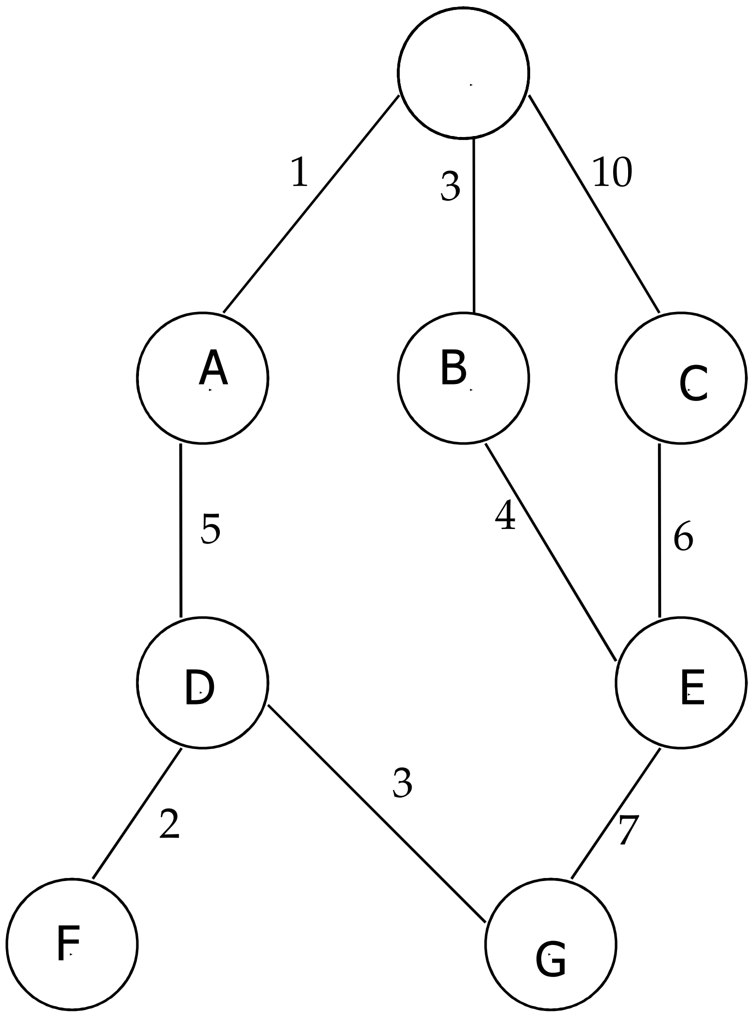
State : $<\#M, \#C, P>$
$\#M$ = Number of missionaries on bank L
$\#C$ = Number of cannibals on bank L
P = Position of the boat

$S0 = <3, 3, L>$
$G = < 0, 0, R >$

<u>Operations</u>
$M2$ = Two missionaries take boat
$M1$ = One missionary takes boat
$C2$ = Two cannibals take boat
$C1$ = One cannibal takes boat
MC = One missionary and one cannibal takes boat

Algorithmics of Search

General Graph search Algorithm



Graph G = (V,E)

1) Open List : $S^{(Ø, 0)}$

Closed list : Ø

2) OL : $A^{(S,1)}$, $B^{(S,3)}$, $C^{(S,10)}$

CL : S

3) OL : $B^{(S,3)}$, $C^{(S,10)}$, $D^{(A,6)}$

CL : S, A

4) OL : $C^{(S,10)}$, $D^{(A,6)}$, $E^{(B,7)}$

CL: S, A, B

5) OL : $D^{(A,6)}$, $E^{(B,7)}$

CL : S, A, B , C

6) OL : $E^{(B,7)}$, $F^{(D,8)}$, $G^{(D, 9)}$

CL : S, A, B, C, D

7) OL : $F^{(D,8)}$, $G^{(D,9)}$

CL : S, A, B, C, D, E

8) OL : $G^{(D,9)}$

CL : S, A, B, C, D, E, F

9) OL : Ø

CL : S, A, B, C, D, E,
F, G

Steps of GGS
(*principles of AI, Nilsson,*)

- 1. Create a search graph *G*, consisting solely of the start node *S*; put *S* on a list called *OPEN.*
- *2.* Create a list called *CLOSED* that is initially empty.
- 3. Loop: if *OPEN* is empty, exit with failure.
- 4. Select the first node on *OPEN*, remove from *OPEN* and put on *CLOSED*, call this node *n*.
- 5. if *n* is the goal node, exit with the solution obtained by tracing a path along the pointers from *n* to *s* in *G*. (ointers are established in step 7).
- 6. Expand node *n*, generating the set *M* of its successors that are not ancestors of *n*. Install these memes of *M* as successors of *n* in *G*.

GGS steps (contd.)

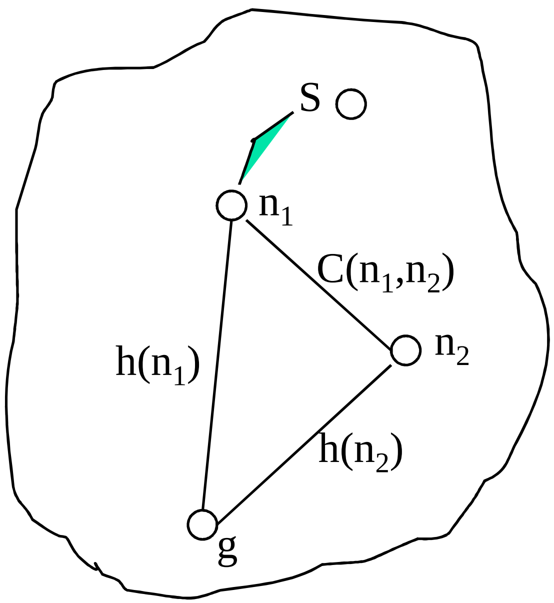
- 7. Establish a pointer to *n* from those members of *M* that were not already in *G* (*i.e.*, not already on either *OPEN* or *CLOSED*). Add these members of *M* to *OPEN*. For each member of *M* that was already on *OPEN* or *CLOSED*, decide whether or not to redirect its pointer to *n*. For each member of M already on *CLOSED*, decide for each of its descendents in *G* whether or not to redirect its pointer.
- 8. Reorder the list *OPEN* using some strategy.
- 9. Go *LOOP.*

GGS is a general umbrella

OL is a queue (BFS)

OL is stack (DFS)

OL is accessed by using a functions *f*= *g*+*h* (Algorithm A)



$$h(n_1) \leq C(n_1, n_2) + h(n_2)$$

Algorithm A

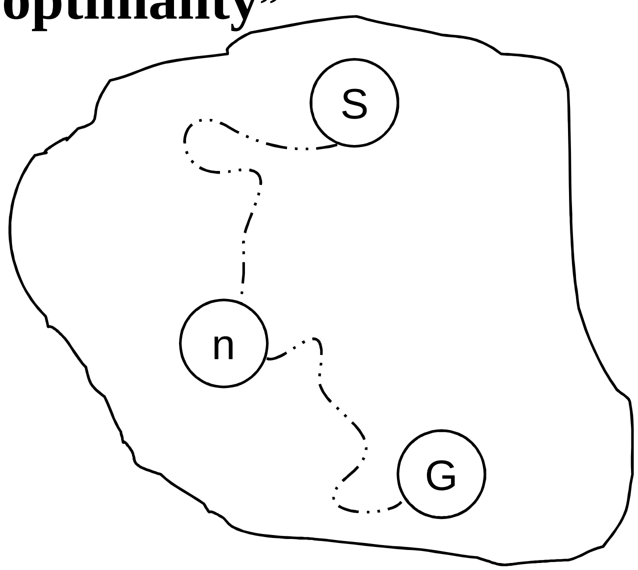
- A function f is maintained with each node

  $f(n) = g(n) + h(n)$, n is the node in the open list
- Node chosen for expansion is the one with least f value
- For BFS: $h = 0$, g = number of edges in the path to S
- For DFS: $h = 0$, $g = \frac{1}{\text{No of edges in the path to S}}$

Algorithm A*

- One of the most important advances in AI
- $g(n)$ = least cost path to n from S found so far
- $h(n) <= h^*(n)$ where $h^*(n)$ is the actual cost of optimal path to G(node to be found) from n

"Optimism leads to optimality"



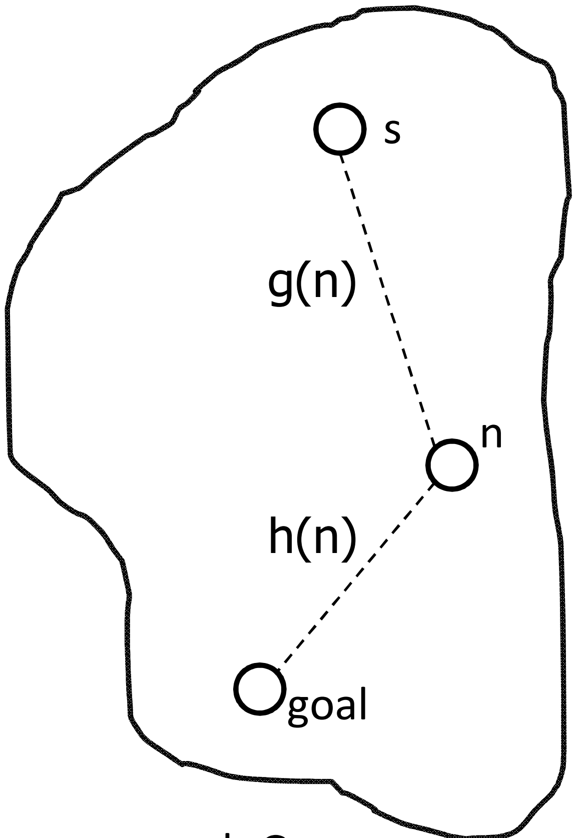
A* Algorithm – Definition and Properties

- $f(n) = g(n) + h(n)$
- The node with the least value of f is chosen from the *OL*.

- $f^*(n) = g^*(n) + h^*(n)$, where,
  $g^*(n)$ = actual cost of the optimal path (s, n)
  $h^*(n)$ = actual cost of optimal path (n, g)

- $g(n) \geq g^*(n)$

- By definition, $h(n) \leq h^*(n)$



State space graph G

8-puzzle: heuristics

Example: 8 puzzle

2	1	4
7	8	3
5	6	

s

1	6	7
4	3	2
5		8

n

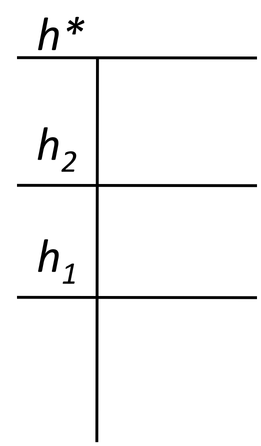
1	2	3
4	5	6
7	8	

g

$h^*(n)$ = actual no. of moves to transform n to g

1. $h_1(n)$ = no. of tiles displaced from their destined position.
2. $h_2(n)$ = sum of Manhattan distances of tiles from their destined position.

$$h_1(n) \leq h^*(n) \text{ and } h_1(n) \leq h^*(n)$$



Comparison

A* Algorithm- Properties

- **Admissibility**: An algorithm is called admissible if it always terminates and terminates in optimal path
- **Theorem**: A* is admissible.
- **Lemma**: Any time before A* terminates there exists on *OL* a node *n* such that $f(n) <= f^*(s)$
- **Observation**: For optimal path $s \rightarrow n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow g$,
  1. $h^*(g) = 0$, $g^*(s)=0$ and
  2. $f^*(s) = f^*(n_1) = f^*(n_2) = f^*(n_3) \ldots = f^*(g)$

A* Properties *(contd.)*

$f^*(n_i) = f^*(s)$, $n_i \neq s$ and $n_i \neq g$

Following set of equations show the above equality:

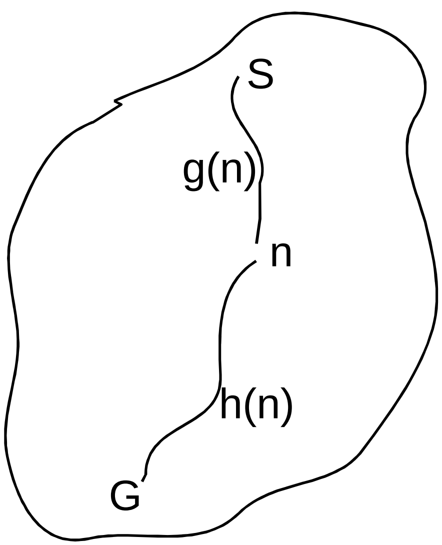
$$f^*(n_i) = g^*(n_i) + h^*(n_i)$$
$$f^*(n_{i+1}) = g^*(n_{i+1}) + h^*(n_{i+1})$$
$$g^*(n_{i+1}) = g^*(n_i) + c(n_i, n_{i+1})$$
$$h^*(n_{i+1}) = h^*(n_i) - c(n_i, n_{i+1})$$

Above equations hold since the path is optimal.

Admissibility of A*

A* always terminates finding an optimal path to the goal if such a path exists.

Intuition



(1) In the open list there always exists a node *n* such that $f(n) <= f^*(S)$.

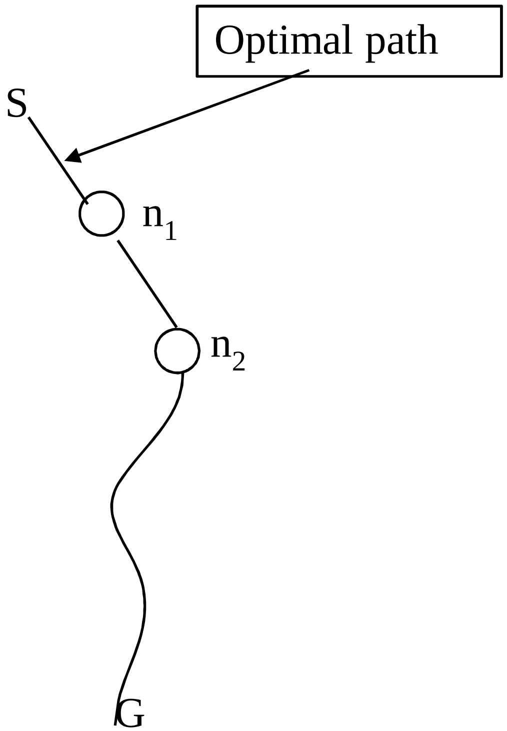
(2) If A* does not terminate, the *f* value of the nodes expanded become unbounded.

1) and 2) are together inconsistent

Hence A* must terminate

Lemma

Any time before A* terminates there exists in the open list a node *n'* such that $f(n') <= f^*(S)$



For any node n_i on optimal path,

$$f(n_i) = g(n_i) + h(n_i)$$
$$<= g^*(n_i) + h^*(n_i)$$

Also $f^*(n_i) = f^*(S)$

Let *n'* be the first node in the optimal path that is in OL. Since <u>all</u> parents of *n'* have gone to CL,

$$g(n') = g^*(n') \text{ and } h(n') <= h^*(n')$$
$$=> f(n') <= f^*(S)$$

If A* does not terminate

Let e be the least cost of all arcs in the search graph.

Then $g(n) >= e.l(n)$ where $l(n)$ = # of arcs in the path from S to n found so far. If A* does not terminate, $g(n)$ and hence $f(n) = g(n) + h(n)$ $[h(n) >= 0]$ will become unbounded.

This is not consistent with the lemma. So A* has to terminate.

2^{nd} part of admissibility of A*

The path formed by A* is optimal when it has terminated

Proof

Suppose the path formed is not optimal

Let G be expanded in a non-optimal path.

At the point of expansion of G,

$f(G) = g(G) + h(G)$

$= g(G) + 0$

$> g^*(G) = g^*(S) + h^*(S)$

$= f^*(S)$ [$f^*(S)$ = cost of optimal path]

This is a contradiction

So path should be optimal

Summary on Admissibility

- 1. A* algorithm halts
- *2.* A* algorithm finds optimal path
- 3. If $f(n) < f^*(S)$ then node n has to be expanded before termination
- 4. If A* does not expand a node n before termination then $f(n) >= f^*(S)$

Better Heuristic Performs Better

Theorem

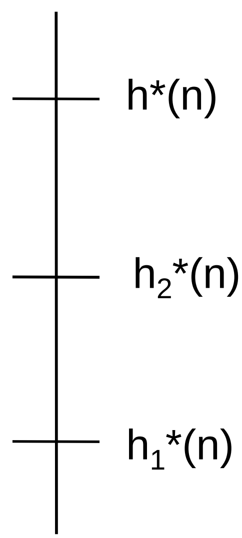
A version A_2* of A* that has a "better" heuristic than another version A_1* of A* performs at least "as well as" A_1*

Meaning of "better"

$h_2(n) > h_1(n)$ for all n

Meaning of "as well as"

A_1* expands at least all the nodes of A_2*



For all nodes n, except the goal node

Proof by induction on the search tree of A_2*.

A* on termination carves out a tree out of *G*

Induction

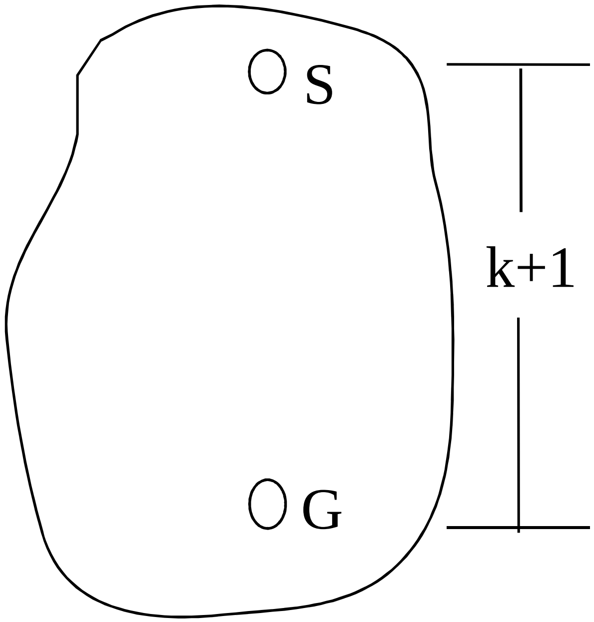
on the depth *k* of the search tree of A_2*. A_1* before termination expands all the nodes of depth *k* in the search tree of A_2*.

k=0. True since start node *S* is expanded by both

Suppose A_1* terminates without expanding a node *n* at depth *(k+1)* of A_2* search tree.

Since A_1* has seen all the parents of *n* seen by A_2*

$g_1(n) <= g_2(n) \qquad (1)$



Since A_1* has terminated without expanding *n*,

$f_1(n) >= f^*(S)$ (2)

Any node whose *f* value is strictly less than $f^*(S)$ has to be expanded.

Since A_2* has expanded *n*

$f_2(n) <= f^*(S)$ (3)

From (1), (2), and (3)

$h_1(n) >= h_2(n)$ which is a contradiction. Therefore, A_1* has to expand all nodes that A_2* has expanded.

<u>Exercise</u>

If better means $h_2(n) > h_1(n)$ for some *n* and $h_2(n) = h_1(n)$ for others, then Can you prove the result ?

Lab assignment

- Implement A* algorithm for the following problems:
  - 8 puzzle
  - Missionaries and Cannibals
  - Robotic Blocks world
- Specifications:
  - Try different heuristics and compare with baseline case, *i.e.*, the breadth first search.
  - Violate the condition $h \leq h^*$. See if the optimal path is still found. Observe the speedup.