CS344: Introduction to Artificial Intelligence (associated lab: CS386)

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Lecture 33: Error Backpropagation

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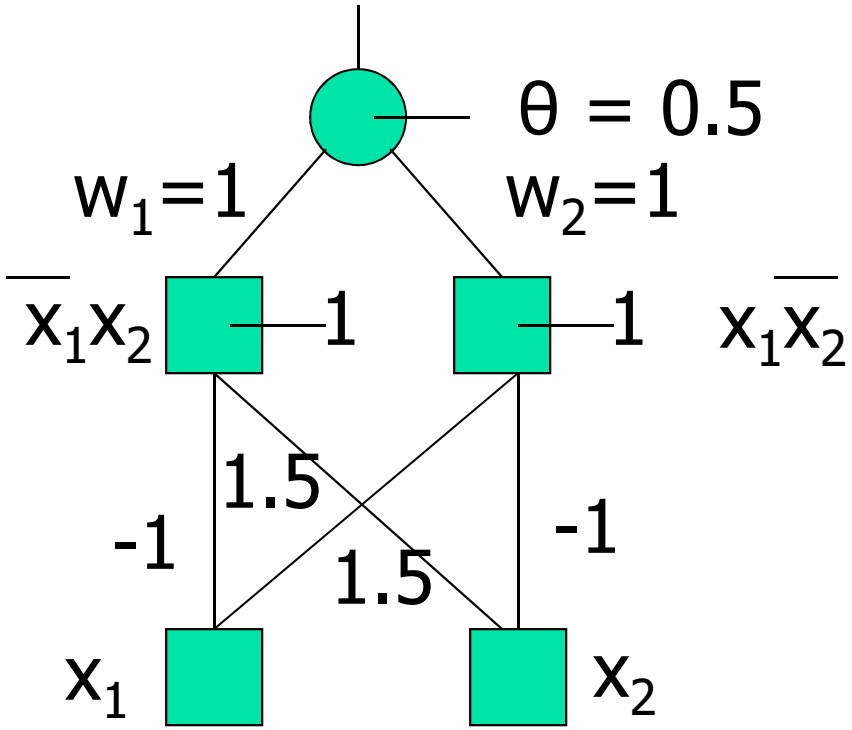
Feedforward Network

Example - XOR



Backpropagation on feedforward network

Gradient Descent Technique

- Let E be the error at the output layer

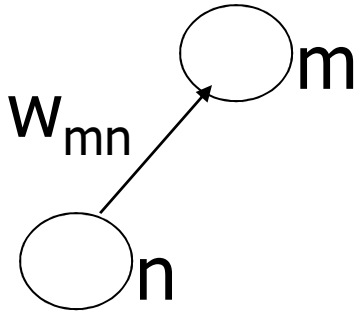
$$E = \frac{1}{2}\sum_{j=1}^{p}\sum_{i=1}^{n}(t_i - o_i)_j^2$$

- t_i = target output; o_i = observed output

- i is the index going over n neurons in the outermost layer
- j is the index going over the p patterns (1 to p)
- Ex: XOR:– p=4 and n=1

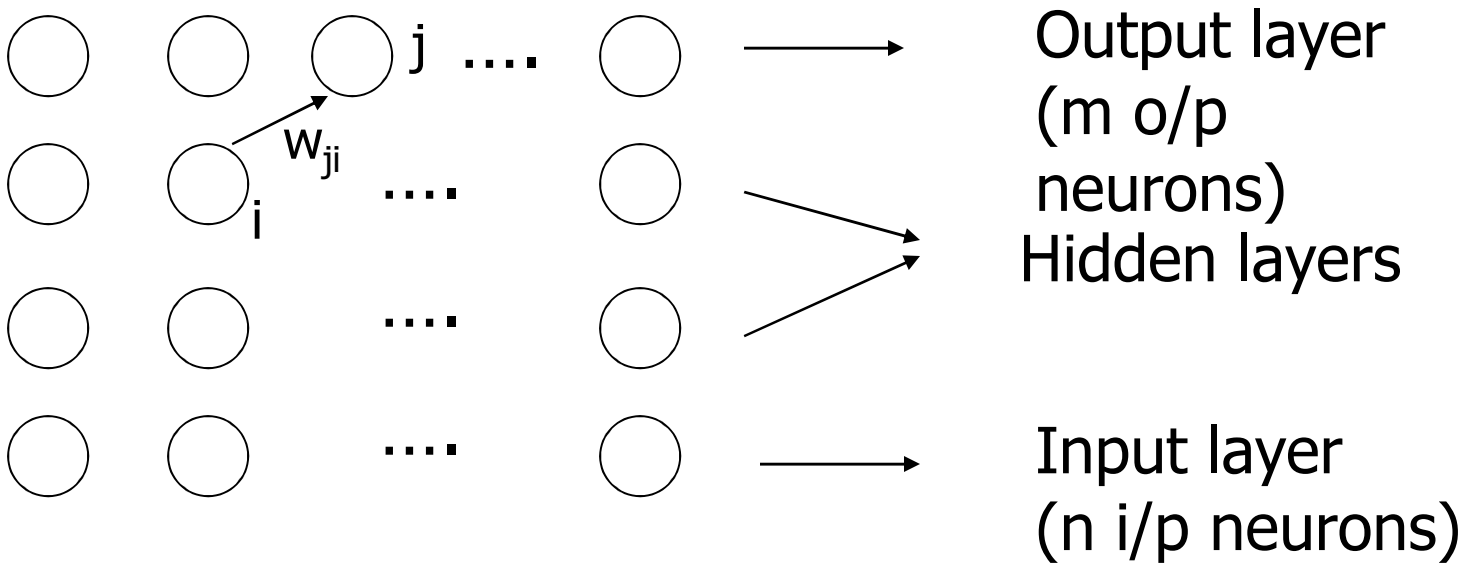
Weights in a FF NN

- w_{mn} is the weight of the connection from the n^{th} neuron to the m^{th} neuron
- E *vs* $\overline{W}$ surface is a complex surface in the space defined by the weights w_{ij}
- $-\frac{\delta E}{\delta w_{mn}}$ gives the direction in which a movement of the operating point in the w_{mn} co-ordinate space will result in maximum decrease in error



$$\Delta w_{mn} \propto -\frac{\delta E}{\delta w_{mn}}$$

Backpropagation algorithm



- Fully connected feed forward network
- Pure FF network (no jumping of connections over layers)

Gradient Descent Equations

$$\Delta w_{ji} = -\eta \frac{\delta E}{\delta w_{ji}} (\eta = \text{learning rate}, 0 \le \eta \le 1)$$

$$\frac{\delta E}{\delta w_{ji}} = \frac{\delta E}{\delta net_j} \times \frac{\delta net_j}{\delta w_{ji}} (net_j = \text{input at the j}^{th} \text{ layer})$$

$$\frac{\delta E}{\delta net_j} = -\delta j$$

$$\Delta w_{ji} = \eta \delta j \frac{\delta net_j}{\delta w_{ji}} = \eta \delta j o_i$$

Backpropagation – for outermost layer

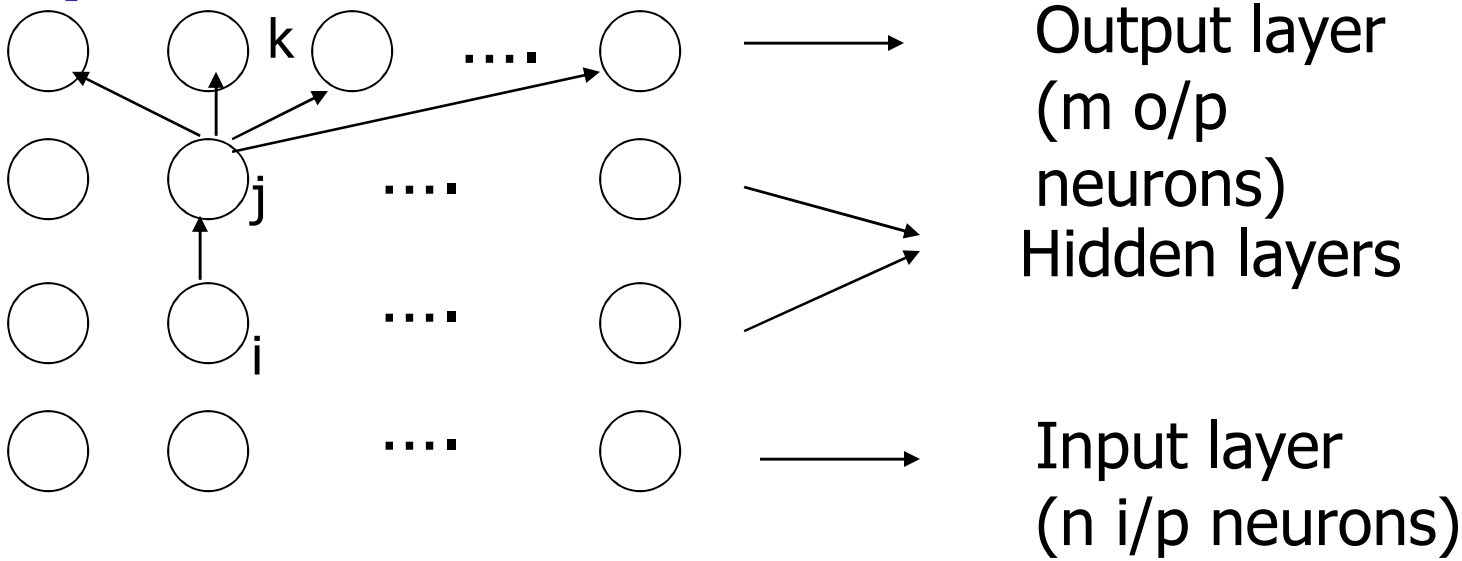
$$\delta j = -\frac{\delta E}{\delta net_j} = -\frac{\delta E}{\delta o_j} \times \frac{\delta o_j}{\delta net_j} \quad (net_j = \text{input at the j}^{th} \text{ layer})$$

$$E = \frac{1}{2}\sum_{p=1}^{m}(t_p - o_p)^2$$

Hence, $\delta j = -(-(t_j - o_j)o_j(1-o_j))$

$$\Delta w_{ji} = \eta(t_j - o_j)o_j(1-o_j)o_i$$

Backpropagation for hidden layers



δ_k is propagated backwards to find value of δ_j

Backpropagation – for hidden layers

$$\Delta w_{ji} = \eta \delta j o_i$$

$$\delta j = -\frac{\delta E}{\delta net_j} = -\frac{\delta E}{\delta o_j} \times \frac{\delta o_j}{\delta net_j}$$

$$= -\frac{\delta E}{\delta o_j} \times o_j (1 - o_j)$$

$$= -\sum_{k \in \text{next layer}} \left(\frac{\delta E}{\delta net_k} \times \frac{\delta net_k}{\delta o_j}\right) \times o_j (1 - o_j)$$

$$\text{Hence, } \delta_j = -\sum_{k \in \text{next layer}} (-\delta_k \times w_{kj}) \times o_j (1 - o_j)$$

$$= \sum_{k \in \text{next layer}} (w_{kj} \delta_k) o_j (1 - o_j)$$

General Backpropagation Rule

- General weight updating rule:

$$\Delta w_{ji} = \eta \delta j o_i$$

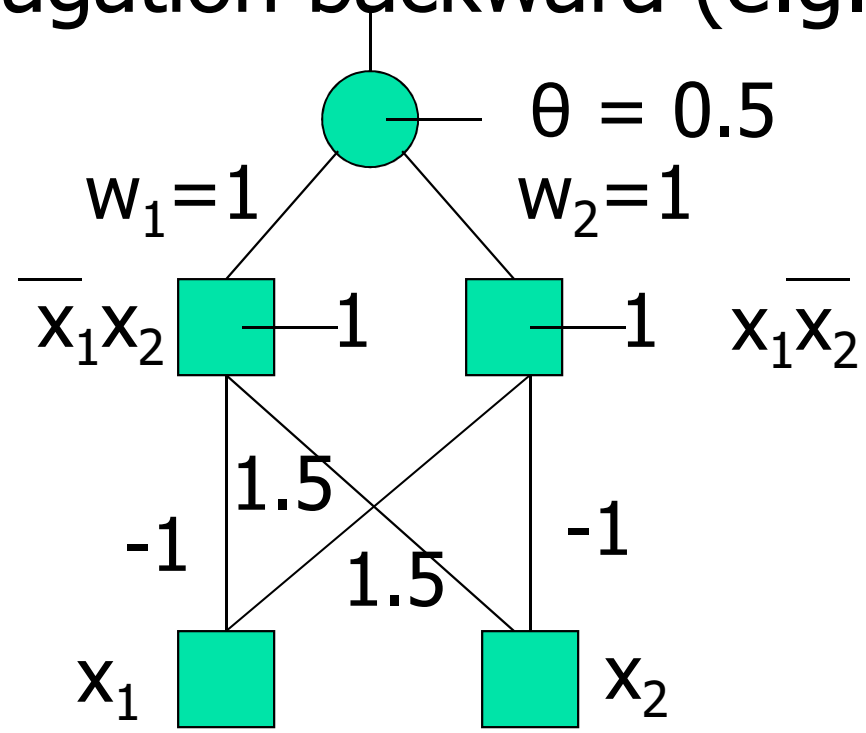
- Where

$$\delta_j = (t_j - o_j) o_j (1 - o_j) \quad \text{for outermost layer}$$

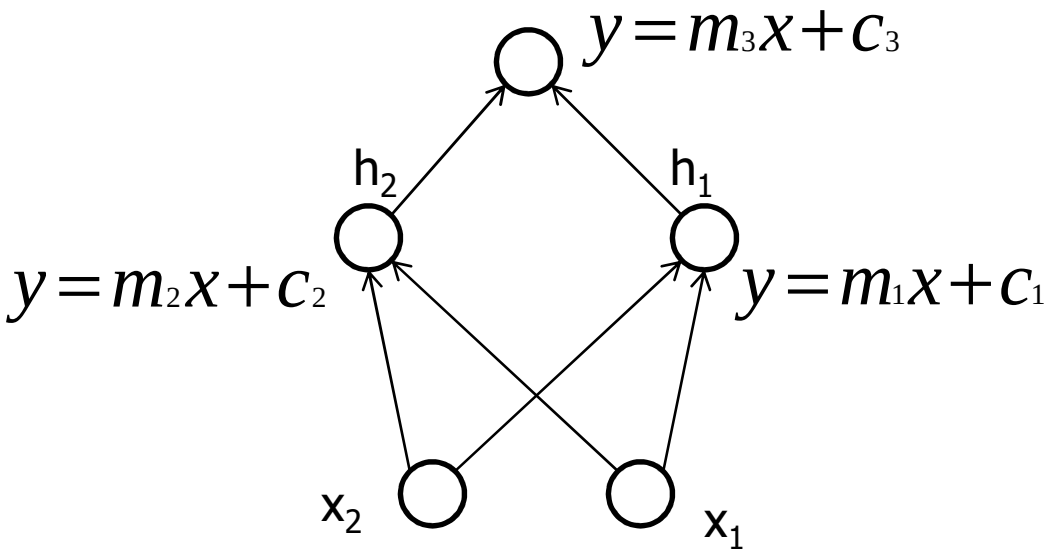
$$= \sum_{k \in \text{next layer}} (w_{kj} \delta_k) o_j (1 - o_j) o_i \quad \text{for hidden layers}$$

How does it work?

- Input propagation forward and error propagation backward (e.g. XOR)



Can Linear Neurons Work?



$$h_1 = m_1(w_1 x_1 + w_2 x_2) + c_1$$

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$$Out = (w_5 h_1 + w_6 h_2) + c_3$$

$$= k_1 x_1 + k_2 x_2 + k_3$$

Note: The whole structure shown in earlier slide is reducible to a single neuron with given behavior

$$Out = k_1 x_1 + k_2 x_2 + k_3$$

Claim: A neuron with linear I-O behavior can't compute X-OR.

Proof: Considering all possible cases:

[assuming 0.1 and 0.9 as the lower and upper thresholds]

For (0,0), Zero class:

$$m(w_1.0 + w_2.0 - \theta) + c < 0.1$$
$$\Rightarrow c - m.\theta < 0.1$$

For (0,1), One class:

$$m(w_2.1 + w_1.0 - \theta) + c > 0.9$$
$$\Rightarrow m.w_1 - m.\theta + c > 0.9$$

For (1,0), One class: $m.w_1 - m.\theta + c > 0.9$

For (1,1), Zero class: $m.w_1 - m.\theta + c > 0.9$

These equations are inconsistent. Hence X-OR can't be computed.

Observations:

1. A linear neuron can't compute X-OR.
2. A multilayer FFN with linear neurons is collapsible to a single linear neuron, hence **no a additional power due to hidden layer.**
3. Non-linearity is essential for power.