CS344: Introduction to Artificial Intelligence (associated lab: CS386)

Pushpak Bhattacharyya
CSE Dept.,
IIT Bombay

Lecture 4: A* and its properties cntd; monotonicity

11th Jan, 2011

Examples

Problem 1 : 8 – puzzle

4	3	6
2	1	8
7		5

S

1	2	3
4	5	6
7	8	

G

Tile movement represented as the movement of the blank space.

Operators:

L : Blank moves left

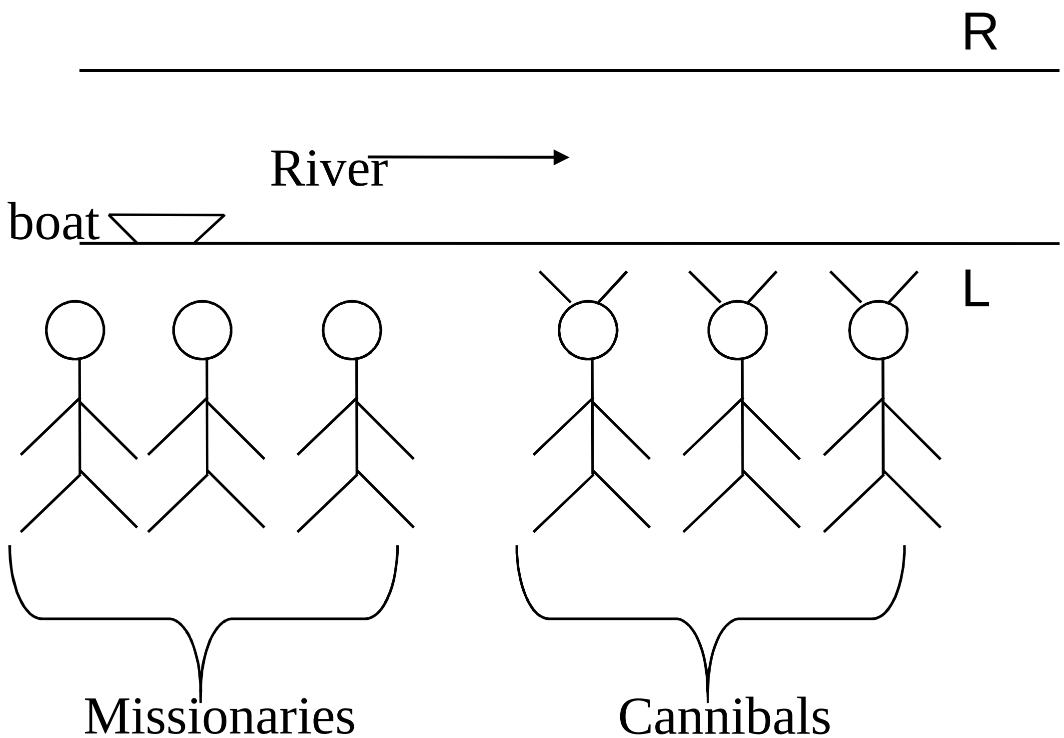
R : Blank moves right

U : Blank moves up

D : Blank moves down

$$C(L) = C(R) = C(U) = C(D) = 1$$

Problem 2: Missionaries and Cannibals



<u>Constraints</u>

- The boat can carry at most 2 people
- On no bank should the cannibals outnumber the missionaries

Steps of GGS
(*principles of AI, Nilsson,)*

- 1. Create a search graph *G*, consisting solely of the start node *S*; put *S* on a list called *OPEN.*
- *2.* Create a list called *CLOSED* that is initially empty.
- 3. Loop: if *OPEN* is empty, exit with failure.
- 4. Select the first node on *OPEN*, remove from *OPEN* and put on *CLOSED*, call this node *n*.
- 5. if *n* is the goal node, exit with the solution obtained by tracing a path along the pointers from *n* to *s* in *G*. (ointers are established in step 7).
- 6. Expand node *n*, generating the set *M* of its successors that are not ancestors of *n*. Install these memes of *M* as successors of *n* in *G*.

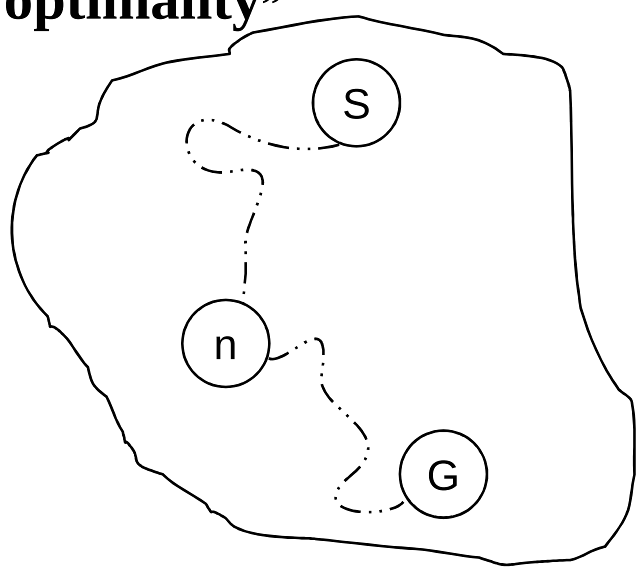
GGS steps (contd.)

- 7. Establish a pointer to *n* from those members of *M* that were not already in *G* (*i.e.*, not already on either *OPEN* or *CLOSED*). Add these members of *M* to *OPEN*. For each member of *M* that was already on *OPEN* or *CLOSED*, decide whether or not to redirect its pointer to *n*. For each member of M already on *CLOSED*, decide for each of its descendents in *G* whether or not to redirect its pointer.
- 8. Reorder the list *OPEN* using some strategy.
- 9. Go *LOOP.*

Algorithm A*

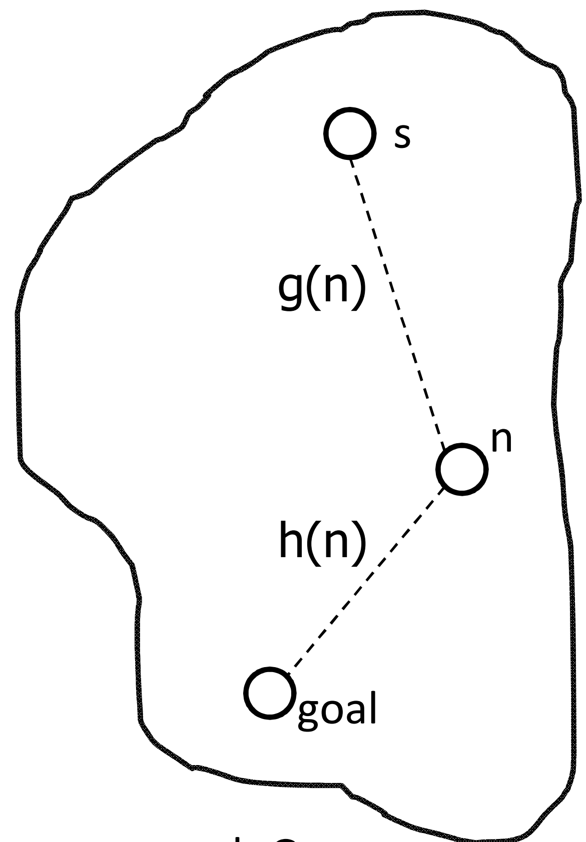
- One of the most important advances in AI
- $g(n)$ = least cost path to n from S found so far
- $h(n) <= h^*(n)$ where $h^*(n)$ is the actual cost of optimal path to G(node to be found) from n

“Optimism leads to optimality”



A* Algorithm – Definition and Properties

- $f(n) = g(n) + h(n)$
- The node with the least value of f is chosen from the *OL*.
- $f^*(n) = g^*(n) + h^*(n)$, where,
  $g^*(n)$ = actual cost of the optimal path (s, n)
  $h^*(n)$ = actual cost of optimal path (n, g)
- $g(n) \geq g^*(n)$
- By definition, $h(n) \leq h^*(n)$



State space graph G

8-puzzle: heuristics

Example: 8 puzzle

2	1	4
7	8	3
5	6	

s

1	6	7
4	3	2
5		8

n

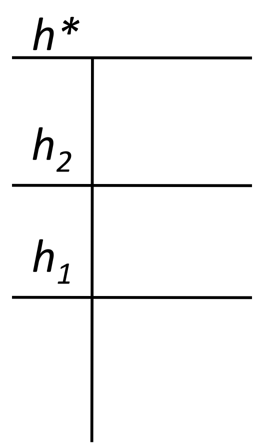
1	2	3
4	5	6
7	8	

g

$h^*(n)$ = actual no. of moves to transform *n* to *g*

1. $h_1(n)$ = no. of tiles displaced from their destined position.
2. $h_2(n)$ = sum of Manhattan distances of tiles from their destined position.

$$h_1(n) \leq h^*(n) \text{ and } h_1(n) \leq h^*(n)$$



Comparison

A* Algorithm- Properties

- **Admissibility**: An algorithm is called admissible if it always terminates and terminates in optimal path
- **Theorem**: A* is admissible.
- **Lemma**: Any time before A* terminates there exists on *OL* a node *n* such that $f(n) <= f^*(s)$
- **Observation**: For optimal path $s \rightarrow n_1 \rightarrow n_2 \rightarrow \ldots \rightarrow g$,
  1. $h^*(g) = 0$, $g^*(s)=0$ and
  2. $f^*(s) = f^*(n_1) = f^*(n_2) = f^*(n_3)\ldots = f^*(g)$

A* Properties *(contd.)*

$f^*(n_i) = f^*(s)$, $\quad n_i \neq s$ and $n_i \neq g$

Following set of equations show the above equality:

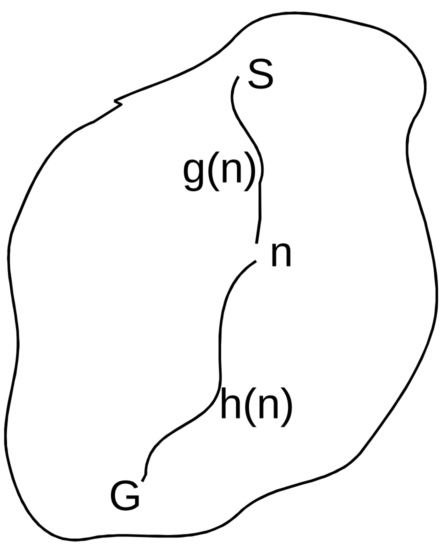
$$f^*(n_i) = g^*(n_i) + h^*(n_i)$$
$$f^*(n_{i+1}) = g^*(n_{i+1}) + h^*(n_{i+1})$$
$$g^*(n_{i+1}) = g^*(n_i) + c(n_i, n_{i+1})$$
$$h^*(n_{i+1}) = h^*(n_i) - c(n_i, n_{i+1})$$

Above equations hold since the path is optimal.

Admissibility of A*

A* always terminates finding an optimal path to the goal if such a path exists.

Intuition



(1) In the open list there always exists a node *n* such that $f(n) <= f^*(S)$.

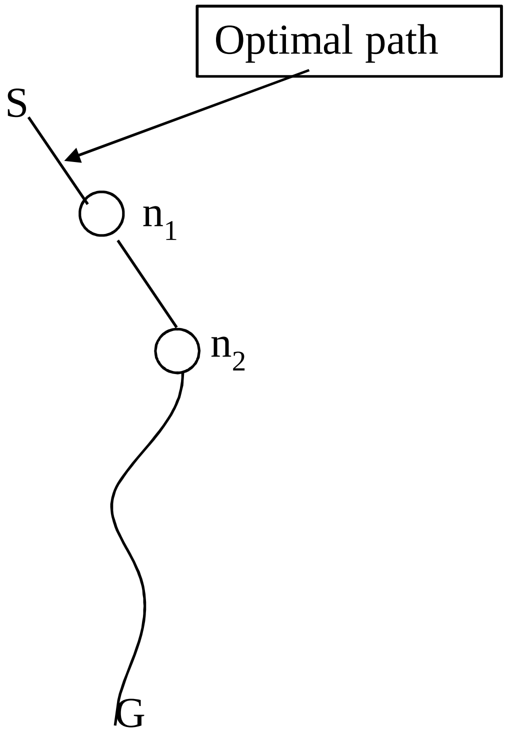
(2) If A* does not terminate, the *f* value of the nodes expanded become unbounded.

1) and 2) are together inconsistent

Hence A* must terminate

Lemma

Any time before A* terminates there exists in the open list a node *n'* such that $f(n') <= f^*(S)$



For any node n_i on optimal path,

$f(n_i) = g(n_i) + h(n_i)$

$\quad <= g^*(n_i) + h^*(n_i)$

Also $f^*(n_i) = f^*(S)$

Let *n'* be the first node in the optimal path that is in OL. Since <u>all</u> parents of *n'* have gone to CL,

$g(n') = g^*(n')$ *and* $h(n') <= h^*(n')$

$=> f(n') <= f^*(S)$

If A* does not terminate

Let *e* be the least cost of all arcs in the search graph.

Then $g(n) >= e.l(n)$ where $l(n)$ = # of arcs in the path from *S* to *n* found so far. If A* does not terminate, $g(n)$ and hence $f(n) = g(n) + h(n)$ $[h(n) >= 0]$ will become unbounded.

This is not consistent with the lemma. So A* has to terminate.

2^{nd} part of admissibility of A*

The path formed by A* is optimal when it has terminated

Proof

Suppose the path formed is not optimal

Let G be expanded in a non-optimal path.

At the point of expansion of G,

$f(G) = g(G) + h(G)$

$= g(G) + 0$

$> g^*(G) = g^*(S) + h^*(S)$

$= f^*(S)$ [$f^*(S)$ = cost of optimal path]

This is a contradiction

So path should be optimal

Summary on Admissibility

- 1. A* algorithm halts
- *2.* A* algorithm finds optimal path
- 3. If $f(n) < f^*(S)$ then node n has to be expanded before termination
- 4. If A* does not expand a node n before termination then $f(n) >= f^*(S)$

Better Heuristic Performs Better

Theorem

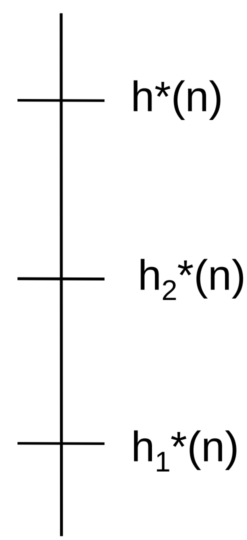
A version A_2* of A* that has a "better" heuristic than another version A_1* of A* performs at least "as well as" A_1*

Meaning of "better"

$h_2(n) > h_1(n)$ for all n

Meaning of "as well as"

A_1* expands at least all the nodes of A_2*



For all nodes n, except the goal node

Proof by induction on the search tree of A_2*.

A* on termination carves out a tree out of *G*

Induction

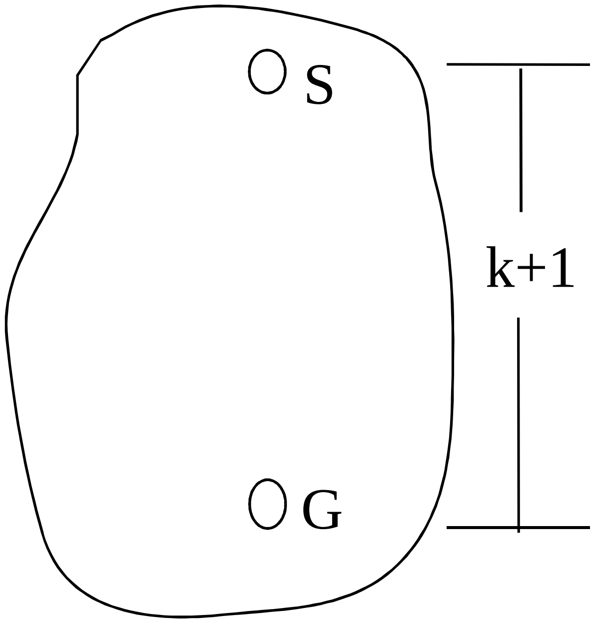
on the depth *k* of the search tree of A_2*. A_1* before termination expands all the nodes of depth *k* in the search tree of A_2*.

k=0. True since start node *S* is expanded by both

Suppose A_1* terminates without expanding a node *n* at depth *(k+1)* of A_2* search tree.

Since A_1* has seen all the parents of *n* seen by A_2*

$g_1(n) <= g_2(n)$ *(1)*



Since A_1* has terminated without expanding *n*,
$f_1(n) >= f^*(S)$ (2)

Any node whose *f* value is strictly less than $f^*(S)$ has to be expanded.
Since A_2* has expanded *n*
$f_2(n) <= f^*(S)$ (3)

From (1), (2), and (3)
$h_1(n) >= h_2(n)$ which is a contradiction. Therefore, A_1* has to expand all nodes that A_2* has expanded.

Exercise

If better means $h_2(n) > h_1(n)$ for some *n* and $h_2(n) = h_1(n)$ for others, then Can you prove the result ?

Lab assignment

- Implement A* algorithm for the following problems:
  - 8 puzzle
  - Missionaries and Cannibals
- Specifications:
  - Try different heuristics and compare with baseline case, *i.e.,* the breadth first search (*h=0*).
  - Violate the condition $h \leq h^*$. See if the optimal path is still found. Observe the speedup.
  - Have as general a program as possible; when a problem is change only a few things should change (say few classes).
  - Present the results in an understandable way, say through graphs and tables.
  - Have enough comments in the code; your marks will be affected by not having enough of this.

Monotonicity

Definition

- A heuristic $h(p)$ is said to satisfy the monotone restriction, if for all 'p', $h(p) <= h(p_c) + cost(p, p_c)$, where '$p_c$' is the child of '$p$'.

Theorem

- If monotone restriction (also called triangular inequality) is satisfied, then for nodes in the closed list, redirection of parent pointer is not necessary. In other words, if any node '*n*' is chosen for expansion from the open list, then $g(n)=g(n^*)$, where $g(n)$ is the cost of the path from the start node '*s*' to '*n*' at that point of the search when '*n*' is chosen, and $g(n^*)$ is the cost of the optimal path from '*s*' to '*n*'

Grounding the Monotone Restriction

7	3	
1	2	4
8	5	6

n

1	2	3
4	5	6
7	8	

g_l

7	3	4
1	2	
8	5	6

n'

h(n) -: number of displaced tiles

Is h(n) monotone ?

$h(n) = 8$

$h(n') = 8$

$C(n,n') = 1$

Hence monotone

Monotonicity of # of Displaced Tile Heuristic

- $h(n) <= h(n') + c(n, n')$
- Any move reduces h(n) by at most 1
- $c = 1$
- Hence, $h(parent) <= h(child) + 1$
- If the empty cell is also included in the cost, then *h* need not be monotone (try!)

Monotonicity of Manhattan Distance Heuristic (1/2)

- *Manhattan distance= X-dist+Y-dist* from the target position
- Refer to the diagram in the first slide:
- $h_{mn}(n) = 1 + 1 + 1 + 2 + 1 + 1 + 2 + 1 = 10$
- $h_{mn}(n') = 1 + 1 + 1 + 3 + 1 + 1 + 2 + 1 = 11$
- $Cost = 1$
- Again, $h(n) <= h(n') + c(n, n')$

Monotonicity of Manhattan Distance Heuristic (2/2)

- Any move can either increase the h value or decrease it by **at most 1.**
- Cost again is 1.
- Hence, this heuristic also satisfies Monotone Restriction
- If empty cell is also included in the cost then manhattan distance does not satisfy monotone restriction (try!)
- Apply this heuristic for Missionaries and Cannibals problem

Relationship between Monotonicity and Admissibility

- Observation:

  Monotone Restriction $\rightarrow$ Admissibility but not vice-versa

- Statement: *If* $h(n_i) <= h(n_j) + c(n_i, n_j)$ *for all* i, j

  then $h(n_i) < = h^*(n_i)$ *for all* i

Proof of Monotonicity→admissibility

Let us consider the following as the optimal path starting with a node $n = n_1 - n_2 - n_3 \ldots n_i - \ldots n_m = g_l$

Observe that

$$h^*(n) = c(n_1, n_2) + c(n_2,n_3) + \ldots + c(n_{m-1}, g_l)$$

Since the path given above is the optimal path from n to g_l

Now,

$h(n_1) <= h(n_2) + c(n_1, n_2)$ ------ Eq 1

$h(n_2) <= h(n_3) + c(n_2, n_3)$ ------ Eq 2

: : : : : :

$h(n_{m-1}) = h(g_i) + c(n_{m-1}, g_i)$------ Eq (m-1)

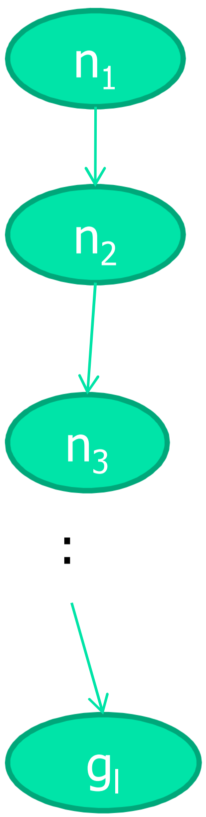
Adding Eq 1 to Eq (m-1) we get

$$h(n) <= h(g_l) + h^*(n) = h^*(n)$$

Hence proved that MR → (h <= h*)

Proof (continued…)

Counter example for vice-versa



$h^*(n_1) = 3$ $\quad$ $h(n_1) = 2.5$

$h^*(n_2) = 2$ $\quad$ $h(n_2) = 1.2$

$h^*(n_3) = 1$ $\quad$ $h(n_3) = 0.5$

: $\quad$: $\quad$: $\quad$:

$h^*(g_l) = 0$ $\quad$ $h(g_l) = 0$

$h < h^*$ everywhere but MR is not satisfied

Proof of MR leading to optimal path for every expanded node (1/2)

Let S-N_1- N_2- N_3- N_4... N_m ...N_k be an optimal path from S to N_k (all of which might or might not have been explored). Let N_m be the **last** node on this path which is on the open list, i.e., *all* the ancestors from S up to N_{m-1} are in the closed list.

For every node N_p on the optimal path,

$g^*(N_p)+h(N_p) <= g^*(N_p)+C(N_p,N_{p+1})+h(N_{p+1})$, by monotone restriction
$g^*(N_p)+h(N_p) <= g^*(N_{p+1})+h(N_{p+1})$ on the optimal path
$g^*(N_m)+ h(N_m) <= g^*(N_k)+ h(N_k)$ by transitivity

Since all ancestors of N_m in the optimal path are in the closed list,

$g(N_m)= g^*(N_m)$.
$=> f(N_m)= g(N_m)+ h(N_m)= g^*(N_m)+ h(N_m) <= g^*(N_k)+ h(N_k)$

Proof of MR leading to optimal path for every expanded node (2/2)

Now if N_k is chosen in preference to N_m,

$$f(N_k) <= f(N_m)$$

$$g(N_k)+ h(N_k) <= g(N_m)+ h(N_m)$$

$$= g^*(N_m)+ h(N_m)$$

$$<= g^*((N_k)+ h(N_k)$$

$$g(N_k)<=g^*(N_k)$$

But $g(N_k)>=g^*(N_k)$, by definition

Hence $g(N_k)=g^*(N_k)$

This means that if N_k is chosen for expansion, the optimal path to this from S has already been found

TRY proving by induction on the length of optimal path

Monotonicity of *f()* values

Statement:

f values of nodes expanded by A* increase monotonically, if *h* is monotone.

Proof:

Suppose n_i and n_j are expanded with temporal sequentiality, *i.e.*, n_j is expanded after n_i

Proof (1/3)…

Possible cases for rigorous proof

n_i expanded before n_j

n_i and n_j co-existing

n_j comes to open list as a result of expanding n_i and is expanded immediately

n_j's parent pointer changes to n_i and expanded

n_j expanded after n_i

Proof (2/3)…

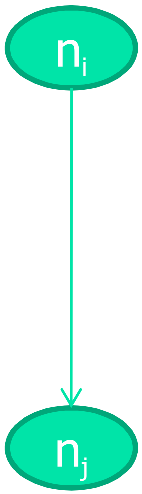
- All the previous cases are forms of the following two cases (think!)
- CASE 1:

  n_j was on open list when n_i was expanded

  Hence, $f(n_i) <= f(n_j)$ by property of A*
- CASE 2:

  n_j comes to open list due to expansion of n_i

Proof (3/3)…



Case 2:

$f(n_i) = g(n_i) + h(n_i)$ (Defn of f)

$f(n_j) = g(n_j) + h(n_j)$ (Defn of f)

$f(n_i) = g(n_i) + h(n_i) = g^*(n_i) + h(n_i)$ ---*Eq 1* (since n_i is picked for expansion n_i is on optimal path)

With the similar argument for n_j we can write the following:

$f(n_j) = g(n_j) + h(n_j) = g^*(n_j) + h(n_j)$ ---*Eq 2*

Also,

$h(n_i) <= h(n_j) + c(n_i, n_j)$ ---*Eq 3* (Parent- child relation)

$g^*(n_j) = g^*(n_i) + c(n_i, n_j)$ ---Eq 4 (both nodes on optimal path)

From *Eq 1, 2, 3 and 4*

$f(n_i) <= f(n_j)$

Hence proved.

Better way to understand monotonicity of *f()*

- Let $f(n_1), f(n_2), f(n_3), f(n_4)\ldots f(n_{k-1}), f(n_k)$ be the f values of k expanded nodes.
- The relationship between two consecutive expansions $f(n_i)$ and $f(n_{i+1})$ nodes always remains the same, *i.e.,* $f(n_i) <= f(n_{i+1})$
- This because
  - $f(n_i)= g(n_i) +h(n_i)$ and
  - $g(n_i)=g^*(n_i)$ since n_i is an expanded node (proved theorem) and this value cannot change
  - $h(n_i)$ value also cannot change Hence nothing after n_{i+1}'s expansion can change the above relationship.

A list of AI Search Algorithms

- A*
  - AO*
  - IDA* (Iterative Deepening)
- Minimax Search on Game Trees
- Viterbi Search on Probabilistic FSA
- Hill Climbing
- Simulated Annealing
- Gradient Descent
- Stack Based Search
- Genetic Algorithms
- Memetic Algorithms