Chapter 3

Linear Algebra

Dixit algorizmi. Or, "So said al-Khwarizmi", being the opening words of a 12^{th} century Latin translation of a work on arithmetic by al-Khwarizmi (*ca.* 780–840).

3.1 Linear Equations

Elementary algebra, using the rules of completion and balancing developed by al-Khwarizmi, allows us to determine the value of an unknown variable x that satisfies an equation like the one below:

$$10x - 5 = 15 + 5x$$

An equation like this that only involves an unknown (like x) and not its higher powers (x^2, x^3), along with additions (or subtractions) of the unknown multiplied by numbers (like $10x$ and $5x$) is called a *linear* equation. We now know, of course, that the equation above can be converted to a special form ("number multiplied by unknown equals number", or $ax = b$, where a and b are numbers):

$$5x = 20$$

Once in this form, it becomes easy to see that $x = b/a = 4$. Linear algebra is, in essence, concerned with the solution of several linear equations in several unknowns. Here is a simple example of two equations and two unknowns x and y, written in a uniform way, with all unknowns (variables) to the left of the equality, and all numbers (constants) to the right:

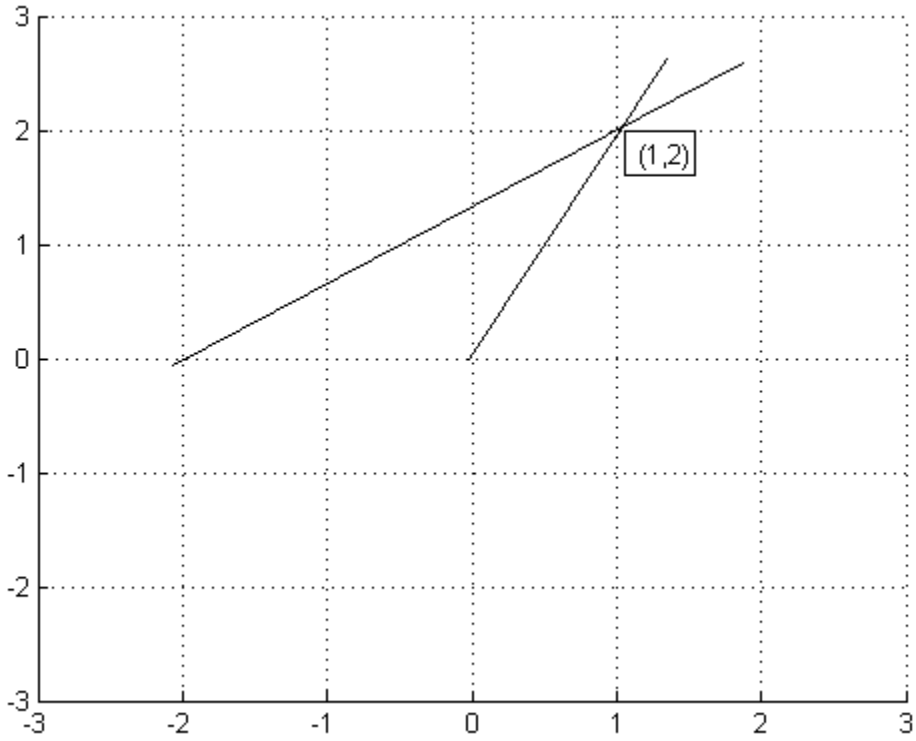

Figure 3.1: Solving linear equations: the geometric view.

$$2x - y = 0$$

$$-x + 2y = 3$$

We woud like to find values of x and y for which these equations are true. School geometry tells us how to visualise this: each equation is a straight line in the xy plane, and since we want a value of x and y for which both equations are true, we are really asking for the values of x and y that lie on both lines (that is, the point of intersection of the two lines: see Fig. 3.1). Of course, if the lines do not meet at a point, then there are no values of x and y that satisfy the equations. And we can continue to solve problems like these geometrically: more unknowns means lines become higher-dimensional flat surfaces ("hyperplanes"), and more equations means we are looking for the single point of intersection of all these surfaces. Visually though, this is challenging for all but a small minority of us, geared as we are to live in a world of three spatial dimensions.

Linear algebra, an extension of elementary algebra, gives us a way of looking at the solution of any number of linear equations, with any number of variables without suffering from this visual overload. In effect, equations are once again converted to the simple form we just saw, that is, $Ax = b$, although A and b are no longer just numbers. In fact, we will see that A is a *matrix*, and that x and b are *vectors* (and in order not to confuse them with variables and numbers, we will from now on use the bold-face notation $\mathbf{x}$ and $\mathbf{b}$). Linear algebra, shows us that solutions, if they exist, can be obtained in three different ways:

1. A direct solution, using techniques called elimination and back substitution.
2. A solution by "inverting" the matrix A, to give the solution $\mathbf{x} = A^{-1}\mathbf{b}$.
3. A vector space solution, by looking at notions called the column space and nullspace of A.

Understanding each of these requires a minimal understanding of vectors and matrices, which we give in a somewhat compressed form here.

3.2 Vectors and Matrices

It is easiest to think of a vector as a generalisation of a single number. A pair of numbers can be represented by a *two-dimensional vector*. Here is the two-dimensional vector representation of the pair $(2, -1)$:

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

This kind of vector is usually called a "column" vector. Geometrically, such a vector is often visualised by an arrow in the two-dimensional plane as shown on the left in Fig. **??**. Multiplying such a vector by any particular number, say 2, multiplies each component by that number. That is, $2\mathbf{u}$ represents the pair $(4, -2)$. Geometrically, we can see that multiplication by a number—sometimes called *scalar multiplication*—simply makes gives a vector with a "longer" arrow as shown on the right in the figure (assuming, of course, that we are not dealing with zero-length vectors). In general, multiplication of a (non-zero) vector $\mathbf{u}$ by different (non-zero) numbers a result in lines either in the direction of $\mathbf{u}$ (if $a > 0$) or in the opposite direction
Suppose we now consider a second vector $\mathbf{v}$ corresponding to the pair $(-1, 2)$, and ask: what is $\mathbf{u} + \mathbf{v}$. This simply adds the individual components. In our example:

$$\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 - 1 \\ -1 + 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Geometrically, the addition of two vectors gives a third, which can visualised as the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ (Fig. **??**, left). It should be straightforward to visualise that any point on the plane containing the vectors $\mathbf{u}$ and $\mathbf{v}$ can be obtained by some linear combination $a\mathbf{u} + b\mathbf{v}$, and that the space of all linear combinations is simply the full two-dimensional plane containing $\mathbf{u}$ and $\mathbf{v}$ (Fig. **??**, right). For the two-dimensional example here, this plane is just the usual xy plane (we will see that this is the vector space $\Re^2$).
Although we have so far only looked at vectors with two components, linear algebra is more general. It allows us to use the same operations with vectors of any size. Suppose our vectors $\mathbf{u}$ and $\mathbf{v}$ are three-dimensional. Linear combinations now still fill a plane containing the two vectors. But, this is no longer the xy plane, since the vectors generated by the linear combinations are points in three-dimensional space (we will see later, that is some "subspace" of the vector space $\Re^3$). Addition of a third vector $\mathbf{w}$ will also not necessarily result in a point on this plane, and the space of linear combinations $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ could fill the entire three-dimensional space.

Let us return now to the two equations that we saw in the previous section:

$$2x - y = 0$$
$$-x + 2y = 3$$

It should be easy to see how these can be written in "vector" form:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{3.1}$$

That is, we are asking if there is some linear combination of the column vectors $[2, -1]$ and $[-1, 2]$ that gives the column vector $[0, 3]$. And this is the point of departure with the usual geometric approach: we visualise solutions of equations not as points of intersections of surfaces, but as linear combination of vectors (of whatever dimension): see Fig. 3.2.

To get it into a form that is even more manageable, we need the concept of a "coefficient matrix". A matrix is simply a rectangular array of numbers, and the coefficient matrix for the left hand side of the linear combination above is:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

This is a 2×2 ("two by two") matrix, meaning it has 2 rows and 2 columns. You can see that the columns of the matrix are simply the column vectors of the linear combination. Let:

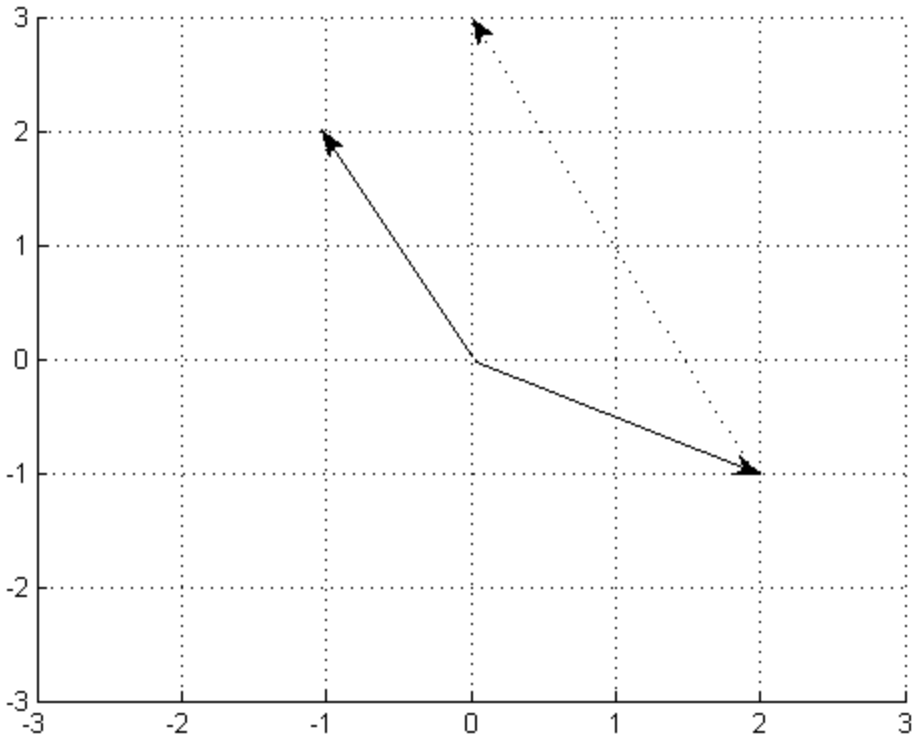

Figure 3.2: Solving linear equations: the geometric view from linear algebra.

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Then, the matrix equation representing the same linear combination is:

$$A\mathbf{x} = \mathbf{b} \tag{3.2}$$

This, as you can see, is just as simple, at least in form. as the very first equation we started with ($5x = 20$). We still need to know what $A\mathbf{x}$ means. Comparing Equations 3.2 and 3.2, $A\mathbf{x} = x$ (column 1 of A) + y (column 2 of A).

This extends easily enough to equations with more variables. Here are three linear equations in three unknowns:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

The coefficient matrix A is:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

The right hand side of the matrix equation is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

What we are trying to do is to find values of x, y and z such that:

$$x(\text{column 1 of } A) + y(\text{column 2 of } A) + z(\text{column 3 of } A) = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

It is easy to see now that the solution we are after is $x = 0, y = 0, z = 1$. Or, in vector form, the solution to the matrix equation $A\mathbf{x} = b$ is:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In general, things are not so obvious and it may be the case that for some values of A and b, no values of x, y and z would solve $A\mathbf{x} = b$. For example, $\mathbf{b}$ may be a point in 3-dimensional space that could not be "reached" by any linear combinations of the vectors comprising the columns of A. Here is a simple example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Seqences of mathematical operations—algorithms, if you will—have been devised to check if solutions exist, and obtain these solutions mechanically when they exist. There are three such approaches we will look at: obtaining solutions by elimination (the simplest), obtaining solutions by matrix inversion (a bit more complex), and finally, a vector space solution (the hardest). We look at each of these in turn.

3.3 Solution of Linear Equations by Elimination

We will now examine a systematic method as "elimination"—first presented by Gauss, for solving a linear system of equations. The basic idea is to progressively *eliminate* variables from equations. For example, let us look once again at the two equations we saw earlier:

$$2x - y = 0$$

$$-x + 2y = 3$$

Elimination first multiplies both sides of the second equation by 2 (this clearly leaves it unchanged):

$$-2x + 4y = 6$$

We can also add equal amounts to the left and right sides without changing the equation. So, adding the left hand side of the first equation to the left hand

side of this new equation, and the right hand side of the first equation to the right hand side of this new equation also does not alter anything:

$$(-2x + 4y) + (2x - y) = 6 + 0 \quad \text{or} \quad 3y = 6$$

So, the two original equations are the same as:

$$\begin{aligned} 2x - y &= 0 \\ 3y &= 6 \end{aligned}$$

You can see that x has been "eliminated" from the second equation and the set of equations have been said to be transformed into an *upper triangular* form. In this form, it is easy to see that $y = 6/3 = 2$. The value of x can then be obtained by substituting back this value for y in the first equation, to give $2x - 2 = 0$ or $x = 1$. The different steps in the elimination process can be expressed clearly using matrices, which we do now. As a running example, we will use the following set of 3 equations:

$$x + 2y + z = 2$$

$$3x + 8y + z = 12$$

$$4y + z = 2$$

We now know what the coefficient matrix for these equations is:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

A point of notation. The entry in row 1, column 1 of A will be denoted a_{11}; row 1, column 2 will be a_{12} and so on. So, in the matrix above, $a_{11} = 1$, $a_{12} = 2$ *etc.*. In general, the entry in row i, column j will be denoted a_{ij}.

Before we plunge into the details of the matrix operations, let us just go through the procedure mechanically (taking on faith for the moment that the steps are indeed valid ones). Our first elimination step is to eliminate x from the second equation. We multiply the first equation by a multiplier and subtract it from the second equation with the goal of eliminating the x coefficient in the second equation. We will call it the $(2, 1)$ step. The first element of the first row a_{11} determines the value of the multiplier (3 in this case) and it is called a *pivot*. For reasons that will become clear, pivots should not be 0. The resultant coefficient matrix is:

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

The next step will be to get a 0 in the first column of the third row (a_{31}) of A_1. Since this is already the case, we do not really need to do anything. But, just to be pedantic, let us take it as giving a coefficient matrix A_2, which is just the same as A_1:

$$A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

We now move on to eliminating a_{32} in A_2. Using a_{22} in A_2 as the next pivot, we subtract from the third row a multiple (2) of the second row. The resultant coefficient matrix is now:

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

A_3 is called an upper triangular matrix for obvious reasons (and sometimes denoted by U). We will see shortly that with the sequence of operations that we have just done, the left hand side of the original matrix equation $A\mathbf{x}$ is transformed into $A_3\mathbf{x}$ by progressively multiplying by a sequence of matrices called "elimination matrices".

3.3.1 Elimination as Matrix Multiplication

Let us go back to the original matrix equation:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

We take a step back and look again at the first elimination step ("multiply equation 1 by 3 and subtract from equation 2"). The effect of this step is to change the right-hand side second equation from 12 to $12 - 3 \times 2 = 6$ and leave the right-hand sides of all other equations unchanged. In matrix notation, the right hand side, after the first elimination step, is:

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$$

A little calculation should be sufficient to convince yourself that $\mathbf{b}_1$ can be obtained by pre-multiplying $\mathbf{b}$ by the matrix:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

That is, $\mathbf{b}_1 = E\mathbf{b}$. You can check this by doing the usual linear combination of the columns of E with the components of $\mathbf{b}$, but the following "row-by-column" view—which is simply the linear combination expanded out—may be even more helpful:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \\ a_{31}b_1 + a_{32}b_2 + a_{33}b_3 \end{bmatrix}$$

So, if the elimination step multiplies the left-hand side of of the matrix equation $A\mathbf{x} = \mathbf{b}$ by the matrix E, then to make sure nothing is changed, we have to do the same to the left-hand side. That is, the elimination step changes the left-hand side to $EA\mathbf{x}$. But now we are stuck—EA is a product of two matrices, which we have not come across before. What does this mean?

Well, we know what we would like EA to mean. We would like $EA = A_1$. That is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \tag{3.3}$$

Taking a vector as simply being a matrix with a single column, we would like to extend the old matrix-vector multiplication ($A\mathbf{x}$) idea to general matrix-matrix multiplication. Suppose B is a matrix comprised of the column vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$. Then AB is a matrix that has columns $A\mathbf{b}_1$, $A\mathbf{b}_2$, and $A\mathbf{b}_3$. So, in the example above, EA is a matrix that has columns $E\mathbf{a}_1$, $E\mathbf{a}_2$ and $E\mathbf{a}_3$ (where $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are the columns of A). Let us work out what these are:

$$E\mathbf{a}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 3 + 0 \times 0 \\ -3 \times 1 + 1 \times 3 + 0 \times 0 \\ 0 \times 1 + 0 \times 3 + 1 \times 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is the first column of the matrix A_1 on the right-hand side of Equation 3.3.1. You can check that $E\mathbf{a}_2$ and $E\mathbf{a}_3$ do indeed give the other two columns of A_1. Once again, there is a "row-by-column" view of multiplying two matrices that you will often find helpful:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}$$

At this point, it is important that you are aware of some properties of matrix multiplication. First, multiplying matrices A and B is only meaningful if the number of columns of A is the same as the number of rows of B. If A is an $m \times n$ matrix, and B is an $n \times k$ matrix, then AB is an $m \times k$ matrix. Second, just like with ordinary numbers, matrix multiplication is "associative"; that is, $(AB)C = A(BC)$ (with numbers, $(3 \times 4) \times 5 = 3 \times (4 \times 5)$. But, unlike ordinary numbers, matrix multiplication is not "commutative". That is $AB \neq BA$ (but with numbers, $3 \times 4 = 4 \times 3$).

It is the associativity of matrix multiplication that allows us to build up a sequence of matrix operations representing elimination. Let us return once again to the matrix equation we started with:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

We have seen, how, by multiplying both sides by the elimination matrix E (which we will now call E_{21}, for reasons that will be obvious), gives:

$$E_{21}(A\mathbf{x}) = (E_{21}A)\mathbf{x} = E_{21}\mathbf{b}$$

or:

$$A_1\mathbf{x} = E_{21}\mathbf{b}$$

where $A_1 = E_{21}A$. Without more elaboration, we now simply present the elimination matrices E_{31} and E_{32} that correspond to the remaining two elimination steps.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

The general rule for constructing an elimination matrix is this. If we are looking at n equations in m unknowns, and an elimination step involves multiplying equation j by a number q and subtracting it from equation i, then the elimination matrix E_{ij} is simply the $n \times m$ "identity matrix" I, with $a_{ij} = 0$ in I replaced by $-q$. For example, with 3 equations in 3 unknowns, and an elimination step that "multiplies equation 2 by 2 and subtracts from equation 3":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Each elimination step therefore results in a multiplication of both sides of $A\mathbf{x} = \mathbf{b}$ by the corresponding elimination matrix. In our example, the three elimination steps give:

$$E_{32}E_{31}E_{21}(A\mathbf{x}) = E_{32}E_{31}E_{21}\mathbf{b}$$

which, using the property of associativity of matrix multiplication is:

$$(E_{32}(E_{31}(E_{21}A)))\mathbf{x} = (E_{32}E_{31}E_{21})\mathbf{b}$$

Or:

$$U\mathbf{x} = (E_{32}E_{31}E_{21})\mathbf{b} = \mathbf{c} \tag{3.4}$$

where U is the upper triangular matrix $E_{32}A_2 = E_{32}(E_{31}A_1 = E_{32}(E_{31}(E_{21}A))$. Here:

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix} \tag{3.5}$$

Before we leave this section, there is one aspect of elimination that we have not yet considered. Let us look at the same equations as before, but in the following order:

$$\begin{aligned} 4y + z &= 2 \\ x + 2y + z &= 2 \\ 3x + 8y + z &= 12 \end{aligned}$$

The coefficient matrix A is then:

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{bmatrix} \tag{3.6}$$

Now clearly, no amount of elimination can get this matrix into an upper triangular form, since that would require a non-zero entry for a_{11}. Since there is no reason to keep the equations in this particular order, we can exchange their order until we reach the one we had in the previous section. Just as a single elimination step can be expressed as multiplication by an elimination matrix, exchange of a pair of equations can be expressed by multiplication by a *permutation* matrix.

The general rule for constructing a permutation matrix is this. If we are looking at m equations in n unknowns, and we want to exchange equations i and j, then the permutation matrix P_{ij} is simply the $m \times n$ "identity matrix" I, with rows i and j swapped:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying a matrix A by P_{12} will swap rows 1 and 2 of A:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 2 & 1 \\ 3 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 3 & 8 & 1 \end{bmatrix}$$

What happens if, in spite of all exchanges, elimination still results in a 0 in any one of the pivot positions? Then we consider the process to have failed, and the equations do not have a solution. Assuming this does not happen, we will reach a point where the original equation $A\mathbf{x} = \mathbf{b}$ is transformed into $U\mathbf{x} = \mathbf{c}$ (as we did in Equation 3.4). The final step is that of *back-substitution*, in which variables are progressively assigned values using the right-hand side of this transformed equation (in Equation 3.3.1, $z = -2$, back-substituted to give $y = 1$, which finally yields $x = 2$).

3.4 Solution of Linear Equations by Matrix Inversion

So, it is possible to represent the steps leading to the solution of a set of linear equations by elimination entirely as a sequence of matrix multiplications. We now look at obtaining the solution by considering matrix "inversion". What we are trying to do is to really find a matrix analog for division with ordinary numbers. There, with an equation like $5x = 20$, we are able to get the answer immediately using division: $x = 20/5$. Can we not do the same with matrices? That is, given $A\mathbf{x} = \mathbf{b}$, can we not get $\mathbf{x} = \mathbf{b}/A$. Well, not quite. But we can get close: we find $\mathbf{x} = A^{-1}\mathbf{b}$, where A^{-1} is the matrix equivalent of $1/A$, and is called the *inverse* of the matrix.

3.4.1 Inverse Matrices

The starting point is just the same as with numbers. We know $a/a = aa^{-1} = 1$ for a non-zero number a. For matrices, we want to find A^{-1} such that $AA^{-1} = I$ where I is the identity matrix. Actually, with matrices, we can ask for inverses in two different ways: AA^{-1} and $A^{-1}A$, called for obvious reasons, right and left inverses of A (recall that since matrix multiplication does not necessarily commute, these could be different).

Let us start with $m \times n$ ("square") matrices. Our definition of an inverse is simply this: if there exists a matrix A_L^{-1} such that $A_L^{-1}A = I$, where I is the $N \times N$ identity matrix, then A_L^{-1} is called the left inverse of A. On the other hand, if there exists a matrix A_R^{-1} such that $AA_R^{-1} = I$, then A_R^{-1} is called the right inverse of A. Now, for square matrices, it is easy to see that the left and right inverses are the same:

$$A_L^{-1}(AA_R^{-1}) = (AA_L^{-1})A_R^{-1}$$

Or,

$$A_L^{-1} = A_R^{-1}$$

So, for square matrices at least, we can simply talk about "the inverse" A^{-1}. The question that now concerns us is: do all square matrices have an inverse? The short answer is "no". Here is a matrix that is not invertible:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \tag{3.7}$$

We can see the conditions under which an inverse exists by referring back to the matrix equation that formed the basis of solution by elimination:

$$A\mathbf{x} = \mathbf{b}$$

Let us assume that A^{-1} exists. Then, the solution reached by elimination would simply be:

$$\mathbf{x} = A^{-1}\mathbf{b} \tag{3.8}$$

Therefore, if the inverse exists, then elimination must produce an upper triangular matrix with non-zero pivots. In fact, the condition works both ways—if elimination produces non-zero pivots then the inverse exists (you can see very quickly that elimination applied to the matrix A in Equation 3.4.1 would give give a row of 0s). Otherwise, the matrix is not invertible, or *singular*. Another way to look at this is that the matrix will be singular if its "determinant" is 0. We will look at what this means later (in Section 3.10), but it is related to the elimination producing non-zero pivots.

If the inverse exists, then the only solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. This gives another way to test for the singularity of a matrix: if there are solutions other than $\mathbf{x} = \mathbf{0}$ to $A\mathbf{x} = \mathbf{0}$. For example, with A in Equation 3.4.1, the vector $\mathbf{x} = [3, -1]$ is a solution to $A\mathbf{x} = \mathbf{0}$.

A final observation may be evident from the example in Equation 3.4.1. A matrix is singular if the columns (or rows) are not linearly independent.

Now let us consider a slight variant of the matrix A in Equation 3.4.1:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

We believe that this matrix is invertible. How can we determine it's inverse? Let the inverse be

$$A^{-1} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{3.9}$$

The system of equations $AA^{-1} = I$ can be written as:

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Again, recall the view of matrix multiplication in which each column on the right hand side is a linear combination of the columns of A:

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, once we solve these two sets of linear equations, we can assemble A^{-1} from the values of a, b, c, and d. We are back, therefore, to solving linear systems of equations— the Gaussian elimination procedure for a single set of linear equations with a single column vector on the right-hand side has to be generalised. The process used is called the *Gauss-Jordan* procedure.

3.4.2 Gauss-Jordan Elimination

The Guass-Jordan elimination method addresses the problem of solving several linear systems $A\mathbf{x}_i = \mathbf{b}_i$ $(1 \leq i \leq N)$ at once, such that each linear system has the same coefficient matrix A but a different right hand side b_i.

From Section 3.3, we know that Gaussian elimination is nothing more than multiplication by elimination matrices, that transforms a coefficient matrix A into an upper-triangular matrix U:

$$U = E_{32}(E_{31}(E_{21}A)) = (E_{32}E_{31}E_{21})A$$

Here E_{ij} is an elimination matrix constructed as we described before (replacing the appropriate 0 in the identity matrix with a non-zero number). Of course, we might, in general, be required to perform row permutation operations and they will simply as appear as multiplication by permutation matrices. But, let us ignore this complication for the moment. Suppose now we applied further elimination steps until U was transformed into the identity matrix. This means multiplication by more matrices:

$$I = E_{13}(E_{12}(E_{23}(E_{32}(E_{31}(E_{21}A))))) = (E_{13}E_{12}E_{23}E_{32}E_{31}E_{21})A = BA \tag{3.10}$$

By definition $B = (E_{13}E_{12}E_{23}E_{32}E_{31}E_{21})$ must be A^{-1}. And this is what Gauss-Jordan does: it simply runs the elimination steps further until the upper-triangular matrix is converted into the identity matrix. So, A^{-1} can be computed by applying the same sequence of elimination steps to the identity matrix. A standard technique for carrying out the same elimination steps on two matrices A and B is to create an augmented matrix $[A\ B]$ and carry out the elimination on this augmented matrix. Gauss-Jordan can therefore be summarised in a single line: perform elimination steps on the augmented matrix $[A\ I]$ (representing the equation $AB = I$) to give the augmented matrix $[I\ A^{-1}]$ (representing the equation $IB = A^{-1}$). Or, in matrix multiplication terms: We illustrate the process with the example matrix we looked at earlier:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \overset{Row_2 - 2\times Row_1}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \overset{Row_1 - 3\times Row_2}{\Longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

One could verify that the inverse of A is given by

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \tag{3.11}$$

Gauss-Jordan therefore gives us a method to construct the inverse of a coefficient matrix A, and therefore directly solve $A\mathbf{x} = \mathbf{b}$ as $\mathbf{x} = A^{-1}\mathbf{b}$.

What if A is not a square matrix but rather a rectangular matrix of size $m \times n$, such that $m \neq n$. Does there exist a notion of A^{-1}? The answer depends on the rank of A.

- If A is full row rank and $n > m$, then AA^T is a full rank $m \times m$ matrix and therefore $(AA^T)^{-1}$ exists. The matrix $A^T(AA^T)^{-1}$ yields the identity matrix when multiplied to A on its right, *i.e.*, $AA^T(AA^T)^{-1} = I$ and is called the right inverse of A. When the right inverse of A is multiplied on its left, we get the projection matrix $A^T(AA^T)^{-1}A$, which projects matrices onto the row space of A.

- If A is full column rank and $m > n$, then A^TA is a full rank $n \times n$ matrix and therefore $(A^TA)^{-1}$ exists. The matrix $(A^TA)^{-1}A^T$ yields the identity matrix when multiplied to A on its left, *i.e.*, $(AA^T)^{-1}A^TA = I$ and is called the left inverse of A. When the left inverse of A is multiplied on its right, we get the projection matrix $A(A^TA)^{-1}A^T$, which projects matrices onto the column space of A.

What if A is neither full row rank nor full column rank? In Section 3.13, we define the pseudoinverse of any $m \times n$ matrix, without any restrictions on its rank.

3.5 Solution of Linear Equations using Vector Spaces

We now turn to the third approach for solving linear equations. This is, in some sense, the most abstract, and involves the idea a vector spaces. A vector space is a collection of vectors that, informally speaking, may be multiplied by a number and added. More formally, a vector space is a set of vectors on which two operations are defined: vector addition and scalar multiplication. Additionally, these two operations satisfy certain natural conditions which we will elaborate shortly. A well-known example is the vector space $\Re^2$. This consists of all 2 dimensional column vectors with real-valued components (this is also just the entire xy plane). Similarly, the space $\Re^n$ comprises all n dimensional column vectors with real-valued components.

More generally, if a set of vectors $\mathcal{V}$ is to qualify as a "vector space" then two vector operations—addition and scalar multiplication—have to be defined, and they have to result in vectors within the set $\mathcal{V}$. The set, or space $\mathcal{V}$ is then said to be "closed" under the operations of addition and scalar multiplication. Now, given vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space, all scalar multiples of vectors $a\mathbf{u}$ and $b\mathbf{v}$ are in the space, as is their sum $a\mathbf{u} + b\mathbf{v}$. That is, all linear combinations of elements in the space are also elements of the space ((V) is closed under linear combination). If a subset (V_S) of any such space is itself a vector space (that is, (V_S) is also closed under linear combination) then (V_S) is called a subspace of (V). All this may seem a bit abstract, and some examples may help:

1. The set of vectors $\Re^2$ consisting of all two-dimensional vectors with real-valued components is a vector space. Adding any two vectors in the set gives a vector that is also in the set. Scalar multiplication of any vector in the set is a vector also in $\Re^2$ (you may find these easy to see by visualising the vectors in the xy plane).

2. The set of vectors $(\Re^2)^+$ consisting of all two-dimensional vectors in the positive quadrant is *not* a vector space. Adding a pair of vectors in $(\Re^2)+$ results in a vector in $(\Re^2)^+$). But, unfortunately, multiplying by a scalar may not. For example, every vector $-3\mathbf{v}$ ($\mathbf{v} \in (\Re^2)^+$) does not belong to $(\Re^2)^+$.

3. The subset $\Re^3_S$ of $\Re^3$ consisting of vectors of any length through the origin $(0, 0, 0)$ is a subspace of $\Re^3$. Adding vectors in $\Re^3_S$ clearly results in an element of the set, as does multiplication by a scalar. It is important that the origin $(0, 0, 0)$ is included: otherwise, multiplication of a vector by 0 would result in a vector not in the subset.

3.5.1 Vector Spaces and Matrices

Our condition on vector spaces has nothing really to do with vectors: all we need is that a pair of operations, addition and scalar multiplication be defined

on a set of elements. So, we are now ready to go a further step, and drop the restriction that vector spaces consist only of vectors. We can, for example, talk of a "vector" space $\mathcal{M}$ consisting of all 2×2 matrices. It is easy to check that this is indeed closed under (matrix) addition and scalar multiplication (we have not come across this before: it is simply the multiplication of every element of the matrix by the number comprising the scalar). Just as with vectors, a subspace of $\mathcal{M}$ is then some subset that is also a vector space.

Vector spaces of matrices provide a novel way of looking at the solution of $A\mathbf{x} = \mathbf{b}$. Recall that $A\mathbf{x}$ is simply a linear combination of the columns of the matrix A. All possible linear combinations of the columns produce a set of all possible column vectors (in effect, all possible $\mathbf{b}$'s). This set is called the *column space* of A, or $C(A)$. Given $\mathbf{b}$, therefore, when we ask: is there a solution to $A\mathbf{x} = \mathbf{b}$, we are really asking if the particular $\mathbf{b}$ we are given is in the column space of A. An example may help. Consider the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

The column space $C(A)$ is a subspace of $\Re^4$ (are you sure you understand why this is so?). We can ask an number of questions now. What is in this subspace? Obviously, each column of A is in $C(A)$. Additionally, $C(A)$ contains all linear combinations of the columns of A. Is $C(A)$ the entire $4-$dimensional space $\Re^4$? If not, how much smaller is $C(A)$ compared to $\Re^4$?

Equivalently, we can pose this problem as follows. Consider the linear system $A\mathbf{x} = \mathbf{b}$. For which right hand sides $\mathbf{b}$ does a solution $\mathbf{x}$ always exist? A solution $\mathbf{x}$ definitely does not exist for every right hand side $\mathbf{b}$, since there are 4 equations in 3 unknowns. Let us analyse this further by writing down the system of equations

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{3.12}$$

Our first point is that there are many vectors $\mathbf{b}$ which cannot be expressed as the linear combination of the columns of A. That leads to the question, *which right hand side* $\mathbf{b}$ *allows the equation to be solved.* One value of $\mathbf{b}$ for which the equation can be solved is the zero vector, for which the corresponding solution is $\mathbf{x} = \mathbf{0}$. Three other trivial values of $\mathbf{b}$ are the values corresponding to every column of A. In particular, we can solve $A\mathbf{x} = \mathbf{b}$ whenever b is in the column

space $C(A)$. When $\mathbf{b}$ is a combination of columns of A, the combination tells us what exactly $\mathbf{x}$ must be.

Do all the columns of A contribute something 'new' to the space $C(A)$[1]? In other words, can we get the same space $C(A)$ using less than three columns of A? In this particular example, the third column of A is a linear combination of the first two columns of A. $C(A)$ is therefor a $2-$dimensional subspace of $\Re^4$. In general, if A is an $m \times n$ matrix, $C(A)$ is a subspace of $\Re^m$.

3.5.2 Null Space

The null space of a matrix A, referred to as $N(A)$, is the space of all solutions to the equation $A\mathbf{x} = 0$. The null space of an $m \times n$ matrix A is a subspace of $\Re^n$.

Consider the example matrix A discussed in the previous section. Its null space is a subspace of $\Re^3$. We will try to figure out the null space of the matrix A by observing the following system of equations:

$$A\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.13}$$

One obvious solution to the system is the zero vector. The null space will always contain the zero vector. Making use of the observation that the columns of A are linearly dependent, we find a second solution to the system as:

$$\mathbf{x}^* = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \tag{3.14}$$

Thus, $\mathbf{x}^*$ is in the null space $N(A)$. Every multiple of $\mathbf{x}^*$ is also in the null space. Each element of the null space $N(A)$ is of the form

$$c.\mathbf{x}^* = \begin{bmatrix} c \\ c \\ -c \end{bmatrix} \tag{3.15}$$

[1] In subsequent sections, we will refer to these columns as *pivot* columns.

where $c \in \Re$. Thus, the null space $N(A)$ is the line passing through the zero vector [0 0 0] and [1 1 -1].

Do solutions to $Ax = 0$ always yield a vector space? The answer is yes and this can be proved by observing that if $A\mathbf{v} = 0$ and $A\mathbf{w} = 0$, then $A(\mathbf{v}+\mathbf{w}) = 0$ and $A(p\mathbf{v}) = 0$ where $p \in \Re$. In general, there are two equivalent ways of specifying a subspace.

1. The first way is to specify a bunch of vectors whose linear combinations will yield the subspace.

2. The second way is to specify a system of equations of the form $A\mathbf{x} = \mathbf{0}$ and any vector $\mathbf{x}$ the satisfies the system is an element of the subspace.

What about the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$ - do elements of this set form a space? The answer is a *no*. An easy way to see this is that the zero vector is not a solution to this system (unless $\mathbf{b}$ is the zero vector) and hence the solutions cannot form a space.

3.6 Elimination for Computing the Null Space (Ax = 0)

In the last section we defined the null space of a matrix A. In this section, we will turn the definition into an algorithm using the elimination technique discussed in Section 3.3. We will take as an example, the following rectangular matrix A

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \tag{3.16}$$

3.6.1 Pivot Variables and Row Echelon Form

We note rightaway that column 2 of A is a multiple of column 1 - it is in the same direction as column 1 and is therefore not indepedent. We expect to discover that in the elimination process for computing the null space $N(A)$. In terms of rows, we observe that the third row is the sum of the first two rows. Thus, the rows are also not independent - and again we hope to discover that in the elimination process.

In essence, what elimination does is change the matrix A and consequently its column space, while leaving the null space of A intact. We first choose the element in position $[1, 1]$ as the pivot element and perform the steps $(2, 1)$ and $(3, 1)$ (recall the definition of a step from Section 3.3) to get the transformed matrix A_1.

$$A_1 = \begin{bmatrix} [1] & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \tag{3.17}$$

We have got the first column in the desirable form. So next, we try to use the element in position $[2, 2]$ as the pivot element. But unfortunately it is a 0. We look below it position $[3, 2]$ hoping for a non-zero element so that we can do a row exachange. But there is a zero below as well! That tells us that second column is dependent on the first column.

Since we have nothing to do with the second column, we move to the thrid column and choose the entry $[2, 3]$ as the pivot. We perform the next elimination step $(3, 2)$, and obtain a third row of zeros. We will denote the resultant matrix by U. Note that the pivots are marked in boxes.

$$U = \begin{bmatrix} [1] & 2 & 2 & 2 \\ 0 & 0 & [2] & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.18}$$

The matrix U is in the row echelon form. A matrix is said to be in row echelon form if it satisfies the following requirements:

1. All nonzero rows are above any rows of all zeroes.
2. The leading coefficient of a row is always strictly to the right of the leading coefficient of the row above it.

While reducing from the matrix A to U, we used only two pivots. In fact, we have already discovered the most important number about the matrix A. The number of pivots is 2 - which is also the *rank* of the matrix.

Fact: *The rank of a matrix is the number of pivots used while reducing it to the row echelon form using elimination.*

We can now solve $U\mathbf{x} = \mathbf{0}$, which has the same solution as $A\mathbf{x} = \mathbf{0}$ (since the elimination steps on zero vector always yield a zero vector). Thus, the null space of U is the same as that of A. How do we describe these solutions? We will first write down the equations corresponding to $U\mathbf{x} = \mathbf{0}$.

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

We will descrie the solution by first separating out the two columns containing the pivots, referred to as *pivot columns* and the remaining columns, referred

to as *free columns*. Variables corresponding to the free columns are called *free variables*, since they can be assigned any value. Variables corresponding to the pivot columns are called *pivot variables*, and their values can be determined based on the values assigned to the free variables. In our example, variables x_2 and x_4 are free variables while x_1 and x_3 are the pivot variables.

Let us say we use the following assignment of values to free variables: $x_2 = 1$, $x_4 = 0$. Then, by back substition, we get the following values: $x_1 = -2$ and $x_3 = 0$. Thus, the following vector $\mathbf{x}'$ is a solution to the system $U\mathbf{x} = \mathbf{0}$ (and consequently the solution to $A\mathbf{x} = \mathbf{0}$) and therefore lies in $N(A)$.

$$\mathbf{x}' = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.19}$$

This solution reiterates our first observation, *viz.*, that column 2 is a multiple of column 1.

We will find some more vectors in the null space. Any multiple $\mathbf{c}.\mathbf{x}', c\Re$ is also in $N(A)$. Note that $\mathbf{c}.\mathbf{x}'$ is a line. Are these the only vectors in $N(A)$? Actually, no – we obtained this set of vectors by assigning only one set of values to the free variables x_2 and x_4. We assign another set of values $x_2 = 0$, $x_4 = 1$, and obtain the values of x_1 and x_3 by back-substitution to get another vector $\mathbf{x}''$ in $N(A)$.

$$\mathbf{x}'' = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \tag{3.20}$$

Now we are in a position to specify all vectors in $N(A)$. The null space will contain all linear combinations of the two special solutions $\mathbf{x}'$ and $\mathbf{x}''$. Every vector in $N(A)$ therefore has the form given in (3.21):

$$ax' + bx'', a \in \Re, b \in \Re \tag{3.21}$$

What is the number of special (linearly independent) solutions for $A\mathbf{x} = \mathbf{0}$ if A is an $m \times n$ matrix? As we saw earlier, the rank r of a matrix equals the number of pivots. The number of free variables is given by

$$no.\ of\ free\ variables\ =\ n -\ r$$

The number of special solutions is exactly equal to the number of free variables. In the above example, we had $n = 4, r = 2$ and therefore number of free variables was 2. The steps for characterizing the null space of a matrix A can be summarized as follows:

1. Reduce A to the row echelon form.
2. If r is the number of pivots, find the $k = n - r$ free variables.
3. Make k different assignments to the free variables. For each assignment, use backsubstitution (using the row echelon form) to find the values of the pivot variables. Each assignemt to the free variables yields a vector in the null space.

3.6.2 Reduced Row Echelon Form

We will take a second look at the matrix U that we obtained by elimination.

$$U = \begin{bmatrix} [1] & 2 & 2 & 2 \\ 0 & 0 & [2] & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.22}$$

The last row of U is composed of zeroes. This is because row 3 of A was a linear combination of rows 1 and 2 and this fact was discovered by elimination. How can we clean U up further? We can do elimination upwards to get zeros above as well as below the pivots. Elimination step $(2, 1)$ on U yields the matrix U'.

$$U' = \begin{bmatrix} [1] & 2 & 0 & -2 \\ 0 & 0 & [2] & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.23}$$

Further, we will make all pivot elements equal to 1 by dividing the corresponding row by the pivot element. This yields the matrix R.

$$R = \begin{bmatrix} [1] & 2 & 0 & -2 \\ 0 & 0 & [1] & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.24}$$

The matrix R has zeroes above and below each pivot. This matrix is called the reduced row echelon form (rref) of A. Matlab has the function *rref(A)* that returns the reduced row echelon form of A. The system of equations $R\mathbf{x} = 0$ is given as

$$x_1 + 2x_2 - 2x_4 = 0$$
$$x_3 + 2x_4 = 0$$

The solution to this system is the same the solution to the original system of equations $Ax = 0$. By simple back-substitution, the vector x can be expressed as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \tag{3.25}$$

Note that the specification of the null space in equation 3.25 is the same as that in equation 3.21.

Let us suppose that we have already got a matrix A in the reduced row echelon form (rref) R. Further, let us pretend that the pivot columns I come before the free columns F. The matrix R can be written as

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \tag{3.26}$$

This block matrix is a very typical rref. We can easily do all the special solutions at once. We will create a *null basis* N whose columns are the special solutions; *i.e.*, $RN = 0$. The following N satisfies this system:

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.27}$$

In fact there is a Matlab command $null(A)$ that returns the null basis of A. It first computes the rref of A and then composes N using the free columns of A as well as the identity matrix of size equal to the rank of A.

Next, we will llustrate the same sequence of steps on the transpose matrix A^t to obtain its row echelon form U and observe the pivots, rank and null space. The solution to $A^t\mathbf{x} = \mathbf{0}$ is a column vector of size 3. Notice that the rank of the transpose is again 2 and is the same as the number of pivot columns. There is a single free column corresponding to the free variable x_3.

$$\begin{bmatrix} [1] & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \overset{E_{2,1},E3,1,E4,1}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \overset{P_{2,3}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & [2] & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \overset{E_{4,2}}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 3 \\ 0 & [2] & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.28}$$

Suppose we make the following assignment to the free variable $x_3 = -c$. Then the solution is given by

$$\begin{bmatrix} -c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{3.29}$$

Thus, the null space of A^t is a line. Taking the elimination steps forward, we can get the reduced row echelon form (as a block matrix) R for matrix A^t.

$$\begin{bmatrix} [1] & 2 & 3 \\ 0 & [2] & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{E_{1,2}}{\Longrightarrow} \begin{bmatrix} [1] & 0 & 1 \\ 0 & [2] & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{(\frac{Row_2}{2})}{\Longrightarrow} \begin{bmatrix} [1] & 0 & 1 \\ 0 & [1] & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{of\ the\ form}{\Longleftrightarrow} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \tag{3.30}$$

The null basis N is

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{3.31}$$

3.6.3 Solving $A\mathbf{x} = \mathbf{b}$

In this sub-section we will illustrate how to completely solve the system $A\mathbf{x} = \mathbf{b}$, if there is a solution. If there is no solution, we will need to identify this fact and if there is some solution, we need to identify how many solutions it has. Our running example will be a system of equations with the same coefficient matrix A that was considered in the previous section (3.6).

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{3.32}$$

The third row is the sum of rows 1 and 2. In other words, if we add the left hand sides, we get the third left hand sides. So we can predict right away what elimination will discover about the right hand side. What is the condition that b_1, b_2 and b_3 satisfy so that there is a solution? Since the sum of the first two left hand sides equals the third left hand side, we require that $b_1 + b_2 = b_3$.

Let us see how elimination discovers this fact. If some combination on the left hand side gives zeros, the same combination on the right hand side should give zeros. Tacking on the vector of $\mathbf{b}$'s as another column to the matrix A, we get the augmented matrix $[A\ \mathbf{b}]$. Applying the elimination steps $E_{2,1}$ and $E_{3,1}$ to the augmented matrix, followed by the elimination step $E_{3,2}$, we get:

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \overset{E_{2,1},E3,1}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\overset{E3,2}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} \tag{3.33}$$

The condition for solvability is therefore $b_3 - b_1 - b_2 = 0$. Thus, the system of equations will have a solution for $\mathbf{b} = [5\ 1\ 6]^T$.

We will now discuss the solvability conditions on the right hand side of a system of equations to ensure that the system of equations $A\mathbf{x} = \mathbf{b}$ is solvable. We will provide a definition in terms of the column space.

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable when $\mathbf{b}$ is in the column space $C(A)$.

Another way of describing solvability is:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable if a combination of the rows of A produces a zero row, the requirement on $\mathbf{b}$ is that the same combination of the components of $\mathbf{b}$ has to yield zero.

It is not immediately apparent that the two systems of equations are equivalent. We will come back to discuss this in a later part of the course. We will proceed to the case when the system of equations does have a solution.

Assuming that the systems of equations $A\mathbf{x} = \mathbf{b}$ is solvable, what is the algorithm (or sequence of steps) to find the complete solution? We will start by finding one particular solution.

1. $\mathbf{x}_{particular}$[2]: Set all free variables (corresponding to columns with no pivots) to 0. In the example above, we should set $x_2 = 0$ and $x_4 = 0$.

2. Solve $A\mathbf{x} = \mathbf{b}$ for pivot variables.

This leaves us with the equations

$$x_1 + 2x_3 = b_1 2x_3 = b_2 - 2b_1$$

Adopting the normal back substitution method, we get

$$x_3 = \frac{b_2 - 2b_1}{2} x_1 = b_2 + 3b_1 \tag{3.34}$$

Thus the particular solution is

$$\mathbf{x}_{particular} = \begin{bmatrix} b_2 + 3b_1 \\ 0 \\ \frac{b_2 - 2b_1}{2} \\ 0 \end{bmatrix}$$

For example, if we choose $\mathbf{b} = [5\ 1\ 6]^T$, we get

$$\mathbf{x}_{particular} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

The sequence of steps is (a) check for solvability conditons (b) substitute some values for the free variables and obtain values for pivot variables. How do we find the complete solution to $A\mathbf{x} = \mathbf{b}$? It is easy to see that any vector $\mathbf{x}_{nullspace}$ in the null space of the matrix A can be added to $\mathbf{x}_{particular}$ and the resultant vector will still remain a solution. Thus, a general solution to the system $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x}_{complete} = \mathbf{x}_{particular} + \mathbf{x}_{nullspace} \tag{3.35}$$

Let us write out the complete solution to this example (recall the null space for this matrix from Equation 3.25).

[2]Since there are many solutions, one could have one's own way of finding one solution.

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3.36}$$

This pattern shows up in all of mathematics, whenever we have linear equations. The reason for this is that

$$A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

In words, this means that if we have one solution, we can add on anything in the null space. This gives us the 'complete' solution. Note that while the null vector can be scaled arbitrarily, the same does not hold for the *particular* solution.

Let us say we want to plot all solutions to this equation. The plot should be in $\Re^4$ because there are 4 unknowns. Does the set of solutions to $A\mathbf{x} = \mathbf{b}$ form a subspace? No, because the space of solutions to this system is not closed under the scaling operation. The null space is a 2-dimensional[3] subspace inside $\Re^4$. The set of solutions to $Ax = b$ does not however pass through the origin, because it must pass through $\mathbf{x}_{particular}$ and then onward. It is like a sub-space shifted away from the origin!

In summary, the algorithm is to go through elimination, find a particular solution and then a special solution. We will now visualize the bigger picture by answering some questions. Consider an $m \times n$ matrix A of rank r.

Q1. What is the relationship between m and r? We know certainly that $r \leq m$ and $r \leq n$. This because, each row as well as column can contain only one pivot and therefore the number of pivots should be less than the number of rows as also less than the number of columns.

Q2. What happens when the rank r is as big as it can be? There are two possibilities here, depending on what the numbers m and n are.

Q3. In the case that A is full column rank, i.e., $r = n$, what can we infer about the null space and the complete solution? Full column rank implies that there is a pivot in every column, that is, there are n pivots. As a result,there are no free variables. The implication is that the null space will only have the $\mathbf{0}$ vector. Therefore, the complete solution is just $\mathbf{x}_{particular}$; there is just one solution, if there is one at all. Thus, the number of solutions is either 0 or 1. There are many applications in reality where the columns are independent and have nothing to look for in the null space, leading to just a particular solution.

We will illustrate by squeezing in an example.

[3]The dimension of the subspace corresponds to the number constants you can choose.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \tag{3.37}$$

The rank of this matrix is 2; elimination will yield exactly 2 pivots. Carrying out the elimination process to the end, we can get the following reduced row echelon form for this matrix:

$$A_{rref} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -17 \\ 0 & -14 \end{bmatrix} \tag{3.38}$$

The first two rows are not dependent, but the other rows are coibinations of the first two rows. It is a case of full column rank. $A\mathbf{x} = \mathbf{b}$ is a system of four equations in two unknowns. If the right hand side is not consistent with the 4 equations, we will get zero solutions. The right hand side $b = [4\ \ 3\ \ 7\ \ 6]^T$ is consistent with the equations and yields one solution. Similarly, the right hand side b which is the sum of the two independent columns of A also gives one unique solution $\mathbf{x} = [1\ 1]^T$. We will maintain the natural symmetry of this discussion by next looking at *full row rank*.

Q4. In the case that A is full row rank, i.e., $r = m$, what can we infer about the null space and the complete solution? Elimination will lead to m pivots; every row will have a pivot. What is the implication on solvability, *i.e.*, for which right hand sides will we have a solution to $A\mathbf{x} = \mathbf{b}$? Since we do not have any zero rows, we can solve the system for every right hand side $\mathbf{b}$. This resolves the question about the existence of a solution. How many free variables will we have? Since $n \geq r$, we will be left with $n - r = n - m$ free variables.

An easy way to obtain an example here (matrix B) is to transpose the above full column rank example matrix A.

$$B = A^T = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \tag{3.39}$$

Elimination yields the following row reduced echelon form with two pivots:

$$\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -11 & -8 \end{bmatrix} \tag{3.40}$$

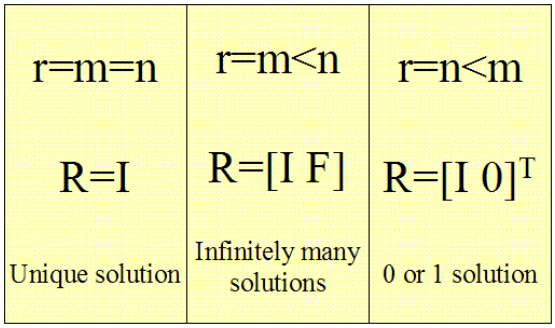

Figure 3.3: Summary of the properties of the solutions to the system of equations $Ax = b$.

The number of free variables is 2.

Q5. In the case that A is full rank, i.e., $r = m = n$, what can we infer about the null space and the complete solution?

This is the most important case of all. We will illustrate with an example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \tag{3.41}$$

The reduced row echelon form for this matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.42}$$

The matrix is invertible; invertible matrices come out naturally in the rref which is the identity matrix. Which are the satisfiable right hand sides $\mathbf{b}$ for the system $A\mathbf{x} = \mathbf{b}$? Since there are no zero rows, there are no constraints on the right hand side. What is the null space of A? It is the zero vector only. Since the rank is also m, the only solution is the *particular solution*, and is therefore a unique solution.

Figure 3.3 summarizes the properties of the solutions to the system of equations $Ax = b$ for different inequality constraints between m, n and r. The rank summarizes the possible solutions, except the exact entries in the solutions.

3.7 Independence, Basis, and Dimension

In this section, we will develop the ideas of linear independence of vectors, the space vectors span, basis for vector spaces and finally the dimension of vector spaces. We will assign clear meanings to these terms.

To set the appropriate background, we will begin with a highly important fact which was mentioned earlier. Let us say we have the system $A\mathbf{x} = 0$, where A is an $m \times n$ matrix and $m < n$. That is, we have more unknowns than equations. The conclusion is that there there are some non-zero vectors in the null space of A. If we perform elimination on A, we will get some pivots and some free columns that do not have pivots because there will be $n - m$ free variables. We can assign non-zero values to the free variables and automatically obtain values for the pivot variables. We will resume from this point.

3.7.1 Independence

Independence *Vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are independent if no linear combination gives the zero vector, except the zero combination. That is,* $\forall c_1, c_2, \ldots, c_n \in \Re$*, such that not all of the* c_i*'s are simultaneously* 0*,* $\sum_i^n c_i \mathbf{x}_i \neq \mathbf{0}$.

For example, in a two dimensional space, a vector $\mathbf{x}$ and twice the vector $2\mathbf{x}$ are dependent, because $(-2) \times \mathbf{x} + (1) \times 2\mathbf{x} = \mathbf{0}$. As another example, suppose we have the vectors $\mathbf{v}_1$ and a zero vector vector $\mathbf{v}_2$, they are dependent because $(0) \times \mathbf{v}_1 + (100) \times \mathbf{v}_2 = \mathbf{0}$.

On the other hand, two non-zero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in a two dimensional space that make an angle $0 < \theta < \frac{\pi}{2}$ with each other will be independent. If we however add a third vector $\mathbf{v}_3$ from the two dimensional space to the set, the three vectors will now be dependent. How do we determine the truth of the above two statements? We could do this as follows. We construct a matrix A with the vectors as three columns $A = [v_1\ v_2\ v_3]$. This matrix is a 2×3 matrix. Does there exist a non-zero solution to the following system?

$$Ac = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.43}$$

It can be easily proved that a non-zero vector $[c_1\ c_2\ c_3]^T$ exists. We will restate the definition for independence in terms of the columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of a matrix A.

Independence *The columns* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *of a matrix* A *are independent if the null-space of* A *is the zero vector. The columns of* A *are dependent only if* $A\mathbf{c} = 0$ *for some* $\mathbf{c} \neq \mathbf{0}$.

In other words, the rank of the matrix A, whose columns are independent is the number of columns n. And in the reduced echelon form, all columns will be pivot columns with no free variables. If the columns are dependent, the rank of A will be less than n, and there will be free variables.

What does it mean for a bunch of vectors to span a space? Recall that we could take all combinations of the columns of a matrix to yield the column space. This column space is the space spanned by the columns of the matrix.

Space spanned by vectors: *Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span a space means that the space consists of all linear combinations of the vectors. Thus, the space spanned by the columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of a matrix A, is the column space of A.*

3.7.2 Basis and Dimension

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, need not be independent in order to span a space. We are specifically interested in a set of vectors that span a space and are at the same time linearly independent. This set of vectors is in some sense, the right number of vectors; even without a single vector from this set, the space cannot be defined. This set is called the *basis*.

Basis for a space: *The basis for a space is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with two properties, viz., (1) The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are independent and (2) These vectors span the space.*

The definition of basis is hinged on the preceeding two definitions - the basis is the set of vectors that is necessary and sufficient for spanning the space. As an example, one basis for the four dimensional space $\Re^4$ is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.44}$$

It is easy to verify that the above vectors are independent; if a combination of the vectors using the scalars in $[c_1, c_2, c_3, c_4]$ should yield the zero vector, we must have $c_1 = c_2 = c_3 = c_4 = 0$. Another way of proving this is by making the four vectors the columns of a matrix. The resultant matrix will be an identity matrix. The null space of an identity matrix is the zero vector. The above basis is often called the *standard basis* for $\Re^4$.

This is not the only basis of $\Re^4$. Consider the following three vectors

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \tag{3.45}$$

These vectors are certainly independent. But they do not span $\Re^4$. This can be proved by showing that the following vector in $\Re^4$ cannot be expressed as a linear combination of these vectors.

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \tag{3.46}$$

In fact, if this vector is added to the set of three vectors in (3.45), together, they define another basis for $\Re^4$. And this could be proved by introducing them as columns of a matrix A, subject A to row reduction and check if there are any free variables (or equivalently, whether all columns are pivot columns). If there are no free variables, we can conclude that the vectors form a basis for $\Re^4$. This is also equivalent to the statement that *if the matrix A is invertible, its columns form a basis for its column space*. This statement can be generalized to $\Re^n$: *if an $n \times n$ matrix A is invertible, its coulumns for a basis for $\Re^n$*.

While there can be many bases for a space, a commonality between all the bases is that they have exactly the same number of vectors. This unique size of the basis is called the dimension of the space.

Dimension: *The number of vectors in any basis of a vector space is called the dimension of the space.*

Do the vectors in (3.45), form a basis for any space at all? The vectors are independent and therefore span the space of all linear combinations of the three vectors. The space spanned by these vectors is a hyperplane in $\Re^4$. Let A be any matrix. By definition, the columns of A span the column space $C(A)$ of A. If there exists a $\mathbf{c} \neq \mathbf{0}$ such that, $A\mathbf{c} = 0$, then the columns of A are not linearly independent. For example, the columns of the matrix A given below are not linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} \tag{3.47}$$

A choice of $\mathbf{c} = [-1\ 0\ 0\ 1]^T$ gives $A\mathbf{c} = 0$. Thus, the columns of A do not form a basis for its columns space. What is a basis for $C(A)$? A most natural choice is the first two columns of A; the thid column is the sum of the first and second columns, while the fourth column is the same as the first column. Also, column elimination[4] on A yields pivots on the first two columns. Thus, a basis for $C(A)$ is

[4]Column elimination operations are very similar to row elimination operations.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \tag{3.48}$$

Another basis for $C(A)$ consists of the first and third columns. We note that the dimension of $C(A)$ is 2. We also note that the rank of A is the number of its pivots columns, which is exactly the dimension of $C(A)$. This gives us a nice theorem.

Theorem 32 *The rank of a matrix is the same as the dimension of its column space. That is, rank*(A) = *dimension* $(C(A))$.

What about the dimension of the null space? We already saw that $\mathbf{c} = [-1\ 0\ 0\ 1]^T$ is in the null space. Another element of the null space is $\mathbf{c}' = [1\ 1\ -1\ 0]^T$. These vectors in the null space specify combinations of the columns that yield zeroes. The two vectors $\mathbf{c}$ and $\mathbf{c}'$ are obviously independent. Do these two vectors span the entire null space? The dimension of the null space is the same as the number of free variables, which happens to be $4 - 2 = 2$ in this example. Thus the two vectors c and c' must indeed span the null space. In fact, it can be proved that the dimension of the null space of an $m \times n$ matrix A is $n - rank(A)$.

The space spanned by the rows of a matrix is called the *row space*. We can also define the row space of a matrix A as the column space of its transpose A^T. Thus the row space of A can be specified as $C(A^T)$. The null space of A, $N(A)$ is often called the *right null space* of A, while the null space of A^T, $N(A^T)$ is often referred to as its *left null space*. How do we visualize these four spaces? $N(A)$ and $C(A^T)$ of an $m \times n$ matrix A are in $\Re^n$, while $C(A)$ and $N(A^T)$ are in $\Re^m$. How can we construct bases for each of the four subspaces? We note that dimensions of $C(A)$ and the rank of $C(A^T)$ should be the same, since row rank of a matrix is its column rank. The bases of $C(A)$ can be obtained as the set of the pivot columns.

Let r be the rank of A. Recall that the null space is constructed by linear combinations of the special solutions of the null space (3.5.2) and there is one special solution for each assignment of the free variables. In fact, the number of special solutions exactly equals the number of free variables, which is $n - r$. Thus, the dimension of $N(A)$ will be $n - r$. Similarly, the dimension of $N(A^T)$ will be $m - r$.

Let us illustrate this on the sample matrix in (3.47).

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} \overset{E_{2,1},E_{3,1}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix} \overset{E_{3,2}}{\Longrightarrow} (R =) \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.49}$$

The reduced matrix R has the same row space as A, by virtue of the nature of row reduction. In fact, the rows of A can be retrieved from the rows of R by reversing the linear operations involved in row elimination. The first two rows give a basis for the row space of A. The dimension of $C(A^T)$ is 2, which is also the rank of A. To find the left null space of A, we look at the system $\mathbf{y}^T A = 0$. Recall the Gauss-Jordan elimination method from Section 3.4.2 that augments A with an $m \times m$ identity matrix, and performs row elimination on the augmented matrix.

$$[A\ I_{m\times m}] \stackrel{rref}{\Longrightarrow} [R\ E_{m\times m}]$$

The rref will consist of the reduced matrix augmented with the elimination matrix reproduced on its right. For the example case in 3.49, we apply the same elimination steps to obtain the matrix E below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{E_{2,1},E_{3,1}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \stackrel{E_{3,2}}{\Longrightarrow} (E =) \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad (3.50)$$

Writing down $EA = R$,

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 3 & 4 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.51)$$

We observe that the last row of E specifies a linear combination of the rows of A that yields a zero vector (corresponding to the last row of R). This is the only vector that yields a zero row in R and is therefore the only element in the basis of the left null space of A, that is, $N(A^T)$. The dimension of $N(A^T)$ is 1.

As another example, consider the space $\mathcal{S}$ of vectors $\mathbf{v} \in \Re^3$ where $\mathbf{v} = [v_1\ v_2\ v_3]^T$ such that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. What is the dimension of this subspace? Note that this subspace is the right null space $N(A)$ of a 1×3 matrix $A = [1\ 1\ 1]$, since $A\mathbf{v} = 0$. The rank, $r = rank(A)$ is 1, implying that the dimension of the right null space is $n - r = 3 - 1 = 2$. One set of basis vectors for $\mathcal{S}$ is $[-1\ 1\ 0]$, $[-1\ 0\ 1]$. The column space $C(A)$ is $\Re^1$ with dimension 1. The left null space $N(A^T)$ is the singleton set $\{0\}$ and as expected, has a dimension of $m - r = 1 - 1 = 0$.

3.8 Matrix Spaces

We will extend the set of examples of vector spaces discussed in Section 3.5 with a new vector space, that of all $m\times n$ matrices with real entries, denoted by $\Re^{m\times n}$.

It is easy to verify that the space of all matrices is closed under operations of addition and scalar multiplication. Additionally, there are interesting subspaces in the entire matrix space $\Re^{m\times n}$, *viz.*,

- set $\mathcal{S}$ of all $n \times n$ symmetric matrices
- set $\mathcal{U}$ of all $n \times n$ upper triangular matrices
- set $\mathcal{L}$ of all $n \times n$ lower triangular matrices
- set $\mathcal{D}$ of all $n \times n$ diagonal matrices

Let $\mathcal{M} = \Re^{3\times 3}$ be the space of all 3×3 matrices. The dimension of $\mathcal{M}$ is 9. Each element of this basis has a 1 in one of the 9 positions and the remaining entries as zeroes. Of these basis elements, three are symmetric (those having a 1 in any of the diagonal positions). These three matrices form the basis for the subspace of diagonal matrices. Six of the nine basis elements of M form the basis of $\mathcal{U}$ while six of them form the basis of $\mathcal{L}$.

The intersection of any two matrix spaces is also a matrix space. For example, $\mathcal{S} \cap \mathcal{U}$ is $\mathcal{D}$, the set of diagonal matrices. However the union of any two matrix spaces need not be a matrix space. For example, $\mathcal{S} \cup \mathcal{U}$ is not a matrix space; the sum $S + U,\ S \in \mathcal{S},\ U \in \mathcal{U}$ need not belong to $\mathcal{S} \cup \mathcal{U}$. We will discuss a special set comprising all linear combinations of the elements of union of two vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ (*i.e.*, $\mathcal{V}_1 \cup \mathcal{V}_2$), and denote this set by $\mathcal{V}_1 \oplus \mathcal{V}_2$. By definition, this set is a vector space. For example, $\mathcal{S} + \mathcal{U} = \mathcal{M}$, which is a vector space.

A property fundamental to many properties of matrices is the expression for a rank 1 matrix. A rank 1 matrix can be expressed as the product of a column vector with a row vector (the row vector forming a basis for the matrix). Thus, any rank 1 matrix X can be expressed as

$$X_{m\times n} = u^T v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \tag{3.52}$$

Let $\mathcal{M}_{m\times n}$ be the set of all $m \times n$ matrices. Is the subset of $\mathcal{M}_{m\times n}$ matrices with rank k, a subspace? For $k = 1$, this space is obviously not a vector space as is evident from the sum of rank 1 matrices, A^1 and B^1, which is not a rank 1 matrix. In fact, the subset of $\mathcal{M}_{m\times n}$ matrices with rank k is not a subspace.

$$A^1 + B^1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 2 \\ 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 3 \\ 4 & 6 & 2 \\ 5 & 6 & 3 \end{bmatrix} \tag{3.53}$$

3.9 Orthogonality and Projection

In this section we will discuss the orthogonality of subspaces. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal *iff*, their dot product is 0. In the eucledian space, the dot product of the two vectors is $x^T y$. The condition $x^T y = 0$ is equivalent to the pythagorous condition between the vectors $\mathbf{x}$ and $\mathbf{y}$ that form the perpendicular sides of a right triangle with the hypotenuse given by $\mathbf{x} + \mathbf{y}$. The *pythagorous condition* is $||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\mathbf{x} + \mathbf{y}||^2$, where the norm is the eucledian norm, given by $||\mathbf{x}||^2 = \mathbf{x}^T\mathbf{x}$. This equivalence can be easily proved and is left to the reader as an exercise. By definition, the vector $\mathbf{0}$ is orthogonal to every other vector.

We will extend the definition of orthogonality to subspaces; a subspace $\mathcal{U}$ is orthogonal to subspace $\mathcal{V}$ *iff*, every vector in $\mathcal{U}$ is orthogonal to every vector in $\mathcal{V}$. As an example:

Theorem 33 *The row space $C(A^T)$ of an $m \times n$ matrix A is orthogonal to its right null space $N(A)$.*

Proof: $A\mathbf{x} = \mathbf{0}$, $\forall \mathbf{x} \in N(A)$. On the other hand, $\forall\ \mathbf{y} \in C(A^T)$, $\exists\ z \in \Re^m$, $\mathbf{s}.t.$, $\mathbf{y} = A^T\mathbf{z}$. Therefore, $\forall\ \mathbf{y} \in C(A^T)$, $\mathbf{x} \in N(A), \mathbf{y}^T\mathbf{x} = \mathbf{z}^T A\mathbf{x} = \mathbf{z}.\mathbf{0} = 0$. □

Not only are $C(A^T)$ and the right null space $N(A)$ orthogonal to each other, but they are also *orthogonal complements* in $\Re^n$, that is, $N(A)$ contains all vectors that are orthogonal to some vector in $C(A^T)$.

Theorem 34 *The null space of A and its row space are orthogonal complements.*

Proof: We note, based on our discussion in Section 3.7.2 that the dimensions of the row space and the (right) null space add up to n, which is the number of columns of A. For any vector $\mathbf{y} \in C(A^T)$, we have $\exists\ \mathbf{z} \in \Re^m$, $\mathbf{s}.t.$, $\mathbf{y} = A^T\mathbf{z}$. Suppose $\forall\ \mathbf{y} \in C(A^T)$, $\mathbf{y}^T\mathbf{x} = 0$. That is, $\forall\ \mathbf{z} \in \Re^m$, $\mathbf{z}^T A\mathbf{x} = 0$. This is possible only if $A\mathbf{x} = \mathbf{0}$. Thus, necessarily, $\mathbf{x} \in N(A)$. □

Along similar lines, we could prove that the column space $C(A)$ and the left null space $N(A^T)$ are orthogonal complements in $\Re^m$. Based on theorem 34, we prove that there is a one-to-one mapping between the elements of row space and column space.

Theorem 35 *If $\mathbf{x} \in C(A^T)$, $\mathbf{y} \in C(A^T)$ and $\mathbf{x} \neq \mathbf{y}$, then, $A\mathbf{x} \neq A\mathbf{y}$.*

Proof: Note that $A\mathbf{x}$ and $A\mathbf{y}$ are both elements of $C(A)$. Next, observe that $\mathbf{x} - \mathbf{y} \in C(A^T)$, which by theorem 34, implies that $\mathbf{x} - \mathbf{y} \notin N(A)$. Therefore, $A\mathbf{x} - A\mathbf{y} \neq \mathbf{0}$ or in other words, $A\mathbf{x} \neq A\mathbf{y}$. □

Similarly, it can be proved that if $\mathbf{x} \in C(A)$, $\mathbf{y} \in C(A)$ and $\mathbf{x} \neq \mathbf{y}$, then, $A^T\mathbf{x} \neq A^T\mathbf{y}$. The two properties together imply a one-to-one mapping between the row and column spaces.

3.9.1 Projection Matrices

The projection of a vector $\mathbf{t}$ on a vector $\mathbf{s}$ is a vector $\mathbf{p} = c\mathbf{s},\ c \in \Re$ (in the same direction as $\mathbf{s}$), such that $\mathbf{t} - c\mathbf{s}$ is orthogonal to $\mathbf{s}$. That is, $\mathbf{s}^T(\mathbf{t} - c\mathbf{s}) = 0$ or $\mathbf{s}^T\mathbf{t} = c\mathbf{s}^T\mathbf{s}$). Thus, the scaling factor c is given by $c = \frac{\mathbf{s}^T\mathbf{t}}{\mathbf{s}^T\mathbf{s}}$. The projection of the vector $\mathbf{t}$ on a vector $\mathbf{s}$ is then

$$\mathbf{p} = \mathbf{s}\frac{\mathbf{t}^T\mathbf{s}}{\mathbf{s}^T\mathbf{s}} \tag{3.54}$$

Using the associative property of matrix multiplication, the expression for $\mathbf{p}$ can be re-written as

$$\mathbf{p} = P\mathbf{t} \tag{3.55}$$

where, $P = \mathbf{s}\mathbf{s}^T\frac{1}{\mathbf{s}^T\mathbf{s}}$ is called the *projection matrix*.

The rank of the projection matrix is 1 (since it is a column mutiplied by a row). The projection matrix is symmetric and its column space is a line through s. For any $d \in \Re$, $P(d\mathbf{s}) = d\mathbf{s}$, that is, the projection of any vector in the direction of $\mathbf{s}$ is the same vector. Thus, $P^2 = P$.

3.9.2 Least Squares

In Section 3.6.3, we saw a method for solving the system $A\mathbf{x} = \mathbf{b}$ (A being an $m \times n$ matrix), when a solution exists. Howevever, a solution may not exist, especially when $m > n$, that is when the number of equations is greater than the number of variables. In Section 3.6.3, we saw that the *rref* looks like $[I\ \mathbf{0}]^T$, where I is an $n \times n$ identity matrix. It could happen that the row reduction yields a zero submatrix in the lower part of A, but the corresponding elements in $\mathbf{b}$ are not zeroes. In other words, $\mathbf{b}$ may not be in the column space of A. In such cases, we are often interested in finding a 'best fit' for the system; a solution $\hat{x}$ that satisfies $A\mathbf{x} = \mathbf{b}$ as well as possible.

We define the best fit in terms of a vector $\mathbf{p}$ which is the projection of $\mathbf{b}$ onto $C(A)$ and solve $A\hat{\mathbf{x}} = \mathbf{p}$. We require that $\mathbf{b} - \mathbf{p}$ is orthogonal to $C(A)$, which means

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \tag{3.56}$$

The vector $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$ is the error vector and is in $N(A^T)$. The equation (3.9.2) can be rewritten as

$$(A^T A)\hat{\mathbf{x}} = A^T \mathbf{b} \tag{3.57}$$

A matrix that plays a key role in this problem is $A^T A$. It is an $n \times n$ symmetric matrix (since $(A^T A)^T = A^T A$). The right null space $N(A^T A)$ is the same as $N(A)$[5]. It naturally follows that the ranks of $A^T A$ and A are the same (since, the sum of the rank and dimension of null space equal n in either case). Thus, $A^T A$ is invertible exactly if $N(A)$ has dimension 0, or equivalently, A is a full column rank.

Theorem 36 *If A is a full column rank matrix (that is, its columns are independent), $A^T A$ is invertible.*

Proof: We will show that the null space of $A^T A$ is $\{0\}$, which implies that the square matrix $A^T A$ is full column (as well as row) rank is invertible. That is, if $A^T A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. Note that if $A^T A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^T A^T A\mathbf{x} = ||A\mathbf{x}|| = 0$ which implies that $A\mathbf{x} = \mathbf{0}$. Since the columns of A are linearly independent, its null space is $\mathbf{0}$ and therefore, $\mathbf{x} = \mathbf{0}$. □

Assuming that A is full column rank, the equation (3.57) can be rewritten as

$$\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}. \tag{3.58}$$

Therefore the expression for the projection $\mathbf{p}$ will be

$$\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b} \tag{3.59}$$

This expression is the n-dimensional equivalent of the one dimensional expression for projection in (3.9.1). The projection matrix in (3.59) is given by $P = A(A^T A)^{-1} A^T$. We will list the solution for some special cases:

- If A is an $n \times n$ square invertible matrix, its column space is the entire $\Re^n$ and the projection matrix will turn out to be the identity matrix.
- Also, if b is in the column space $C(A)$, then $\mathbf{b} = A\mathbf{t}$ for some $t\ in \Re^n$ and consequently, $P\mathbf{b} = A(A^T A)^{-1}(A^T A)\mathbf{t} = A\mathbf{t} = \mathbf{b}$.

[5]The proof is left as an exercise.

- On the other hand, if b is orthogonal to $C(A)$, it will lie in $N(A^T)$, and therefore, $A^T\mathbf{b} = 0$, implying that $\mathbf{p} = 0$.

Another equivalent way of looking at the best fit solution $\hat{\mathbf{x}}$ is a solution that minimizes the square of the norm of the error vector

$$e(\hat{\mathbf{x}}) = ||A\mathbf{x} - \mathbf{b}||^2 \tag{3.60}$$

Setting $\frac{de(\hat{\mathbf{x}})}{d\mathbf{x}} = 0$, we get the same expression for $\hat{\mathbf{x}}$ as in (3.9.2). The solution in 3.9.2 is therefore often called the *least squares solution*. Thus, we saw two views of finding a best fit; first was the view of projecting into the column space while the second concerned itself with minimizing the norm squared of the error vector.

We will take an example. Consider the data matrix A and the coefficient matrix $\mathbf{b}$ as in (3.61).

$$Ax = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \tag{3.61}$$

The matrix A is full column rank and therefore A^TA will be invertible. The matrix A^TA is given as

$$A^TA = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

Substituting the value of A^TA in the system of equations (3.57), we get,

$$6\hat{x}_1 - 3\hat{x}_2 = 2 \tag{3.62}$$
$$-3\hat{x}_1 + 6\hat{x}_2 = 8 \tag{3.63}$$

The solution of which is, $x_1 = \frac{4}{5}$, $x_2 = \frac{26}{15}$.

3.9.3 Orthonormal Vectors

A collection of vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ is said to be orthonormal *iff* the following condition holds $\forall\ i, j$:

$$\mathbf{q}_i^T \mathbf{q}_j \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{3.64}$$

A large part of numerical linear algebra is built around working with orthonormal matrices, since they do not overflow or underflow. Let Q be a matrix comprising the columns $\mathbf{q}_1$ through $\mathbf{q}_n$. It can be easily shown that

$$Q^T Q = I_{n \times n}$$

When Q is square, $Q^{-1} = Q^T$. Some examples of matrices with orthonormal columns are:

$$Q_{rotation} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}, \; Q_{reflection} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix},$$
$$Q_{Hadamard} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \; Q_{rect} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.65}$$

The matrix $Q_{rotation}$ when multiplied to a vector, rotates it by an angle θ, whereas $Q_{reflection}$ reflects the vector at an angle of $\theta/2$. These matrices present standard varieties of linear transformation, but in general, premultiplication by an $m \times n$ matrix transforms from an input space in $\Re^m$ to an input space in $\Re^n$. The matrix $Q_{Hadamard}$ is an orthonormal matrix consisting of only 1's and -1's. Matrices of this form exist only for specific dimensions such as 2, 4, 8, 16, *etc.*, and are called *Hadamard matrices*[6]. The matrix Q_{rect} is an example rectangular matrix whose columns are orthonormal.

Suppose a matrix Q has orthonormal columns. What happens when we project any vector onto the column space of Q? Substituting $A = Q$ in (3.59), we get[7]:

$$\mathbf{p} = Q(Q^T Q)^{-1} Q^T \mathbf{b} = Q Q^T \mathbf{b} \tag{3.66}$$

Making the same substitution in (3.9.2),

$$\hat{\mathbf{x}} = (A^T Q)^{-1} Q^T \mathbf{b} = Q^T \mathbf{b} \tag{3.67}$$

[6] An exhaustive listing of different types of matrices can be found at `http://en.wikipedia.org/wiki/List_of_matrices`.

[7] Note that $Q^T Q = I$. However, $QQ^T = I$ only if Q is a square matrix.

The i^{th} component of $\mathbf{x}$, is given by $x_i = q_i^T b$.

Let Q_1 be one orthonormal basis and Q_2 be another orthonormal basis for the same space. Let A be the coefficient matrix for a set of points represented using Q_1 and B be the coefficient matrix for the same set of points represented using Q_2. Then $Q_1 A = Q_2 B$, which implies that B can be computed as $B = Q_2^T Q_1 A$. This gives us the formula for changing basis.

3.9.4 Gram-Schmidt Orthonormalization

The goal of the Gram-Schmidt orthonormalization process is to generate a set of orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, given a set of independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. The first step in this process is to generate a set of orthogonal vectors $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ from $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. To start with, $\mathbf{t}_1$ is chosen to be $\mathbf{a}_1$. Next, the vector $\mathbf{t}_2$ is obtained by removing the projection of $\mathbf{a}_2$ on $\mathbf{t}_1$, from $\mathbf{a}_2$, based on (3.9.1). That is,

$$\mathbf{t}_2 = \mathbf{a}_2 - \frac{1}{\mathbf{a}_1^T \mathbf{a}_1} \mathbf{a}_1 \mathbf{a}_1^T \mathbf{a}_2 \tag{3.68}$$

This is carried out iteratively for $i = 1, 2, \ldots, n$, using the expression below:

$$\mathbf{t}_i = \mathbf{a}_i - \frac{1}{\mathbf{t}_1^T \mathbf{t}_1} \mathbf{t}_1 \mathbf{t}_1^T \mathbf{a}_i - \frac{1}{\mathbf{t}_2^T \mathbf{t}_2} \mathbf{t}_2 \mathbf{t}_2^T \mathbf{a}_i - \ldots - \frac{1}{\mathbf{t}_{i-1}^T \mathbf{t}_{i-1}} \mathbf{t}_{i-1} \mathbf{t}_{i-1}^T \mathbf{a}_i \tag{3.69}$$

This gives us the orthogonal vectors $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$. Finally, the orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are obtained by the simple expression

$$\mathbf{q}_i = \frac{1}{||\mathbf{t}_i||} \mathbf{t}_i \tag{3.70}$$

Let A be the matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and Q, the matrix with columns $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$. It can be proved that $C(V) = C(Q)$, that is, the matrices V and Q have the same column space. The vector a_i can be expressed as

$$\mathbf{a}_i = \sum_{k=1}^{n} (\mathbf{a}_i^T \mathbf{q}_k) \mathbf{q}_k \tag{3.71}$$

The i^{th} column of A is a linear combination of the columns of Q, with the scalar coefficient $\mathbf{a}_i^T \mathbf{q}_k$ for the k^{th} column of Q. By the very construction procedure of the Gram-Schmidt orthonormalization process, $\mathbf{a}_i$ is orthogonal to $\mathbf{q}_k$ for all $k > i$. Therefore, (3.71) can be expressed more precisely as

$$\mathbf{a}_i = \sum_{k=1}^{i} (\mathbf{a}_i^T \mathbf{q}_k)\mathbf{q}_k \tag{3.72}$$

Therefore, matrix A can be decomposed into the product of Q with a upper triangular matrix R; $A = QR$, with $R_{k,i} = \mathbf{a}_i^T \mathbf{q}_k$. Since $\mathbf{a}_i^T \mathbf{q}_k = 0,\ \forall\ k > i$, we can easily see that R is upper traingular.

3.9.5 Fourier and Wavelet Basis

The hermetian inner product of a complex vector $\mathbf{x}$ with another complex vector $\mathbf{y}$ is $\overline{\mathbf{x}}^T\mathbf{y}$, and is also denoted by $\mathbf{x}^H\mathbf{y}$. A complex matrix Q is called orthonormal if $\overline{Q}^T Q = I$. Consider the complex number $c = cos(\frac{2\pi}{n}) + icos(\frac{2\pi}{n})$. Then, $c^n = 1$. The *fourier matrix* F_n is defined as

$$F_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & c & \dots & c^{n-1} \\ . & . & \dots & . \\ 1 & c^{k-1} & \dots & c^{(k-1)(n-1)} \\ . & . & \dots & , \\ 1 & c^{n-1} & \dots & c^{(n-1)(n-1)} \end{bmatrix} \tag{3.73}$$

The (hermetian) inner products of distinct columns F_n are 0, while the inner product of a column with itself is 1. Therefore, the columns of F_n are orthonormal and form a basis for $\Re^n$. Consequently, the inverse of F_n is its conjugate transpose $\overline{F}_n^T$.

Further, $F_{2^k}, k \geq 1$ can expressed as

$$F_{2^k} == \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{2^{k-1}} & 0_{2^{k-1}} \\ 0_{2^{k-1}} & F_{2^{k-1}} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}}_{P} \tag{3.74}$$

where, $D = diag(1, c, c^2, \ldots, c^{2^k})$ and 0_{2^k} is a $2^k \times 2^k$ matrix of 0's. This factorization, applied recursively, can reduce the time for computing F_{2^k} from $\mathcal{O}(n^2)$ to $\mathcal{O}(n \log n)$. This is the idea behind the *fast fourier transform algorithm*. Though the factorization discussed here, applies to only to powers of 2, there exist FFT algorithms [**?**] for any number (including primes).

An advantage of representing a vector in $\Re^n$ (for example, a $\sqrt{n} \times \sqrt{n}$ subblock of an image matrix) using the fourier basis is that certain basis components of the representation could be ignored to achieve minimally lossy compression of matrices such as image matrices. Another orthogonal basis that is used for minimally lossy matrix compression is the wavelet basis. A sample wavelet basis matrix W for $\Re^8$ is

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.75}$$

The discrete wavelet transform can be computed efficiently using a *fast wavelet transform algorithm* which is less computationally complex, taking $\mathcal{O}(n)$ time as compared to $\mathcal{O}(nlogn)$ for the fast fourier transform.

3.10 Determinants

Every square matrix A has a real number associated with it, called its determinant and it is denoted by $det(A)$. In this sequel, we will often refer to the following $n \times n$ matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ a_{k1} & a_{k2} & \ldots & a_{kn} \\ . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{3.76}$$

We will describe four fundamental properties of the determinant, which essentially define the determinant.

1. The determinant of the identity matrix I is 1. That is, $det(I) = 1$.

2. When two rows of A are permuted (*c.f.* Section 3.3.1), the sign of the determinant changes. That is $det\left(Perm(A, j, k)\right) = -det(A)$, where $Perm(A, j, k)$ returns a matrix formed by exchanging the j^{th} and k^{th} rows of A for any $1 \leq j, k \leq n$.

3. If any row of A is scaled by $t \in \Re$, the determinant also gets scaled by t. Thus, if

$$S(A, k, t) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ ta_{k1} & ta_{k2} & \dots & ta_{kn} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3.77}$$

   then

$$det\left(S(A, k, t)\right) = t \times det(A) \tag{3.78}$$

   The function $S(A, k, t)$ returns a matrix eaxactly with all the entries of A, except for the k^{th} row, which is scaled by $t \in \Re$.

4. The sum of the determinants of two $n \times n$ matrices, A and B, with all $(n-1)$ rows the same, except for the k^{th} row, $1 \leq k \leq n$, equals the determinant of an $n \times n$ matrix C that has the same $n-1$ rows from A/B, but with the k^{th} row being the sum of the k^{th} rows of A and B. Thus, if

$$B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ b_{k1} & b_{k2} & \dots & b_{kn} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3.79}$$

   and,

$$C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{k1} + b_{k1} & a_{k2} + b_{k2} & \dots & a_{kn} + b_{kn} \\ . & . & \dots & , \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3.80}$$

then

$$det(C) = det(A) + det(B)$$

Using these basic properties of determinants, we infer some **derived properties**:

1. If a matrix A has two equal rows, its determinant must be 0.

   Proof: Let B be the matrix obtained by permuting the two equal rows of A. By the second property of determinants, $det(B) = -det(A)$. Since the permuted rows are the same, $B = A$, which implies that $det(B) = det(A)$. The two equalities on determinants of A and B imply that $det(A) = 0$. □

2. The determinant of A obtained by subtracting $\rho \in \Re$ times the j^{th} row from the k^{th} row leaves the determinant unaltered. Therefore, if

$$E = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{k1} - \rho a_{j1} & a_{k2} - \rho a_{j2} & \dots & a_{kn} - \rho a_{jn} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3.81}$$

   we will have

$$det(E) = det(A)$$

   Proof:

$$det(E) = det(A) + det\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ -\rho a_{j1} & -\rho a_{j2} & \dots & -\rho a_{jn} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\right)$$

$$= det(A) - \rho \times det\left(\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\right) = det(A) \quad (3.82)$$

The first step follows from the fourth fundamental property of determinants, the second follows from the third fundamental property, while the third step is a consequence of the first derived property of determinants. □

This property shows that the row elimination steps discussed in Section 3.3.1, leave the determinant unchanged.

3. If any k^{th} row of A is 0, then $det(A) = 0$ *Proof:* Consider a matrix A' that has the same rows as A for all $1 \leq i \leq n$, except for the $i = k$. Let the k^{th} row of A' be the same as its j^{th} row, for some $1 \leq j \leq n$, such that $j \neq k$. Note that by the first derived property, $det(A') = 0$. The matrix A can be obtained from A' by subtracting the j^{th} row of A' from its k^{th} row. By the second derived property, $det(A) = det(A')$. Thus, $det(A) = 0$. Another simpler proof is that $S(A, k, 0) = A$ which implies that $det(A) = det\left(S(A, k, 0)\right) = 0 \times det(A) = 0$ (by third fundamental property of determinants). □

4. The determinant of an upper triangular matrix U is the product of its diagonal entries.

   Proof: Row elimination operations can be performed on an upper traingular matrix U to yield a diagonal matrix D (*c.f.* Section 3.4.2 on Gauss-Jordan elimination), while neither performing any row exchanges nor altering the diagonal entries of U. By the second derived property of determinants, $det(D) = det(U)$. Using the first fundamental property of determinants, $det(D)$ can be proved to be the product of its diagonal entries, which is also the product of the diagonal entries of U. □

   In fact, most mathematical softwares compute the determinant of a matrix A by first reducing it to an upper triangular matrix U by row elimination

on A (which preserves the determinant, by virtue of the second derived property) and then compute the product of the diagonal entries (which also happen to be the pivots) of U. If some α row exchanges are performed during the reduction of A to U, the product of the diagonal entries, multiplied by $(-1)^{\alpha}$ yields the determinant of A. As an example, consider the 2×2 matrix A_2:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{3.83}$$

Using the derived property (2), the matrix $A_2^{'}$ can proved to have same determinant as A.

$$A_2^{'} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21} * a_{12}}{a_1 1} \end{bmatrix} \tag{3.84}$$

$A_2^{'}$ is an upper triangular matrix with $det(A_2^{'})$ given by

$$det(A_2^{'}) = a_{11}a_{22} - a_{21} * a_{12} \tag{3.85}$$

Therefore,

$$det(A) = a_{11}a_{22} - a_{21} * a_{12} \tag{3.86}$$

5. The determinant of a matrix A is 0 *iff* A is singular (or non-invertible).

   Proof Sketch: We will consider the proof in two parts. When A is singular, elimination, with some possible permutations, yields (as discussed in Section 3.4.1) an upper traingular matrix with some diagonal (pivot) entries that are 0s. Therefore, by the derived property (4), $det(A) = 0$. When A is non-singular, elimination yields an upper triangular matrxi with no zero entries. Consequently, we will have $det(A) \neq 0$. □

6. The determinant of the product of two matrices A and B is the product of their determinants[8], *i.e.*, $det(AB) = det(A) \times det(B)$.

[8] Recall that determinant does not have the linear additive property.

A corollary of this property is that $det(A^{-1}) = \frac{1}{det(A)}$ because $det(A^{-1})det(A) = det(I) = 1$. Similarly, $det(A^n) = (det(A))^n$ and $det(A_1 A_2 \dots A_n) = det(A_1)det(A_2) \dots det(A_n)$.

In this context, it will be appropriate to point out that determinants also have relationship with volumes of solids. The determinant of matrix A in (3.76), is the volume of an $n-$dimensional parallelotope, with corners at $(0, 0, \dots, 0)$, $(a_{11}, a_{12}, \dots, a_{1n})$, ..., $(a_{n1}, a_{n2}, \dots, a_{nn})$. The parallelotope corresponding to $I_{n\times n}$ is an $n-$dimensional unit hypercube in n dimensions and has a volume of 1. An orthonormal matrix Q represents a hypercube in n dimensions and has volume given by $det(Q) = \sqrt{det(I)} = 1$. If Q is orthogonal (and not necessarily orthonormal), its volume is $\prod_{i=1}^{n} s_i$, where s_i is the factor by which the i^{th} row of Q should be scaled, so that the row has unit norm. Determinants make easy the task of computing areas of parallelotopes. If the parallelotope does not have any corner at the origins, the coordinates of the corners can be computed relative any one of the corners and the area can be computed using determinants.

7. The determinant of a square matrix A equals the determinant of its transpose, *i.e.*, $det(A) = det(A^T)$.

   Proof Sketch: We can decompose A as a LU, where L is a lower traingular matrix and U is an upper traingular matrix. That is $A = LU$. Consequently, $A^T = U^T L^T$. By derived property (6), $det(A) = det(L)det(U)$ and $det(A^T) = det(U^T)det(L^T)$. Since the diagonal entries of L^T and U^T are the same as the diagonal entries of L and U respectively, and since derived property (4) states that the determinants of L, L^T, U and U^T are just products of their respective diagonal entries, we have $det(A) = det(A^T)$. □

   By virtue of this property, all the properties of determinants discussed so far with respect to scaling or exchanging rows hold for similar manipulations on the columns, since column operations on A are row operations on A^T.

3.10.1 Formula for determinant

In (3.86), we showed the formula for the determinant of a 2×2 matrix A_2. The formula can also be obtained by using the basic property (4), decomposing $det(A_2)$ into the sum of determinants of 4 matrices, with one surviving element per row. We will use the notation $|.|$ instead of $det\,([.])$ to denote the determinant of a matrix.

$$det(A_2) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{3.87}$$

Of these, there are only two nonzero terms; the terms with zero columns or zero rows have 0 determinant. The determinant of a 3×3 matrix can be similarly computed, by decomposing the determinant as the sum of $3 \times 3 \times 3 = 27$ determinants. However, many of the determinants in the sum turn out to be 0, either because of zero rows or zero columns. Each of the non-zero terms have exactly one entry for each row and each column. Thus, the determinant of a 3×3 matrix can be expressed as

$$\begin{aligned} det(A_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} \\ &+ \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \\ &+ \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \\ &= a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{cofactor\ of\ a_{11}} + a_{12}\underbrace{(-1)\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{minor\ of\ a_{12}}}_{cofactor\ of\ a_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{cofactor\ of\ a_{13}} \end{aligned} \tag{3.88}$$

In (3.88), the determinant of A_3 is decomposed into the sum of signed determinants of smaller 2×2 matrices called *co-factors*, each scaled by a corresponding *factor*. The sign of the co-factors depend on the number of row permutations required to get the matrix in a diagonal form; the sign is $(-1)^{num\ perms}$ which happens to be $(-1)^{i+j}$. In general, for an $n \times n$ matrix A, the *minor* of a term a_{ij} is the determinant of an $(n-1) \times (n-1)$ sub-matrix of A that has the row i and column j removed, while its *co-factor* is the minor multiplied by $(-1)^{i+j}$. The minor for a_{ij} is denoted by M_{ij} and its co-factor by C_{ij}. Minors of the form M_{ii} are called principal minors.

The general formula for the determinant of an $n \times n$ matrix contains $n!$ terms, corresponding to all permutations of the choice of the column index for the non-zero entries corresponding to each row index. That is,

$$det(A) = \sum_{(p_1,p_2,\ldots,p_n) \in Perm(1,2,\ldots,n)} a_{(1,p_1)} a_{(2,p_2)} \ldots a_{(n,p_n)} \tag{3.89}$$

In terms of co-factors, the formula for determinant is

$$det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} \tag{3.90}$$

for any $1 \leq i \leq n$.

3.10.2 Formula for Inverse

Let A be an $n \times n$ invertible matrix. In Section 3.4.1, we saw an elegant algorithm for computing A^{-1}. In (3.91), we present a closed form expression for A^{-1} in terms of the co-factors of A, even though the expression is very expensive to compute.

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ . & . & \ldots & . \\ C_{k1} & C_{k2} & \ldots & C_{kn} \\ . & . & \ldots & . \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}^T = \frac{1}{det(A)} C^T \tag{3.91}$$

We denote the matrix in (3.91) consisting of the co-factors of A, by C. It can be easily verified that the expression in (3.91) is indeed A^{-1}.

$$AA^{-1} = \frac{1}{det(A)} \begin{bmatrix} \sum_{j=1}^{n} a_{1j}C_{1j} & \sum_{j=1}^{n} a_{1j}C_{2j} & \dots & \sum_{j=1}^{n} a_{1j}C_{nj} \\ \sum_{j=1}^{n} a_{2j}C_{1j} & \sum_{j=1}^{n} a_{2j}C_{2j} & \dots & \sum_{j=1}^{n} a_{2j}C_{nj} \\ . & . & \dots & . \\ . & . & \sum_{j=1}^{n} a_{ij}C_{ij} & . \\ . & . & \dots & . \\ \sum_{j=1}^{n} a_{nj}C_{1j} & \sum_{j=1}^{n} a_{nj}C_{2j} & \dots & \sum_{j=1}^{n} a_{nj}C_{nj} \end{bmatrix}$$

$$= \frac{1}{det(A)} \begin{bmatrix} det(A) & 0 & \dots & 0 \\ 0 & det(A) & \dots & 0 \\ . & . & \dots & . \\ . & . & det(A) & . \\ . & . & \dots & . \\ 0 & 0 & \dots & det(A) \end{bmatrix} = I_{n\times n} \quad (3.92)$$

Recall from (3.90) that $det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}$ for any $1 \leq i \leq n$. However, $\sum_{j=1}^{n} a_{ij}C_{kj} = 0$, if $i \neq k$. This is because, for $i \neq k$, $\sum_{j=1}^{n} a_{ij}C_{kj}$ is the determinant of a matrix that has identical i^{th} and k^{th} rows and hence equals 0, by the derived property (1) for matrices.

The formula (3.91) for matrix inverse (if it exists) can be substituted in (3.4.1) to yield the *Cramer's rule* for solving the system $A\mathbf{x} = \mathbf{b}$. The Cramer's rule is:

$$x = A^{-1}\mathbf{b} = \frac{1}{det(A)} \begin{bmatrix} \sum_{j=1}^{n} b_j C_{1j} \\ \sum_{j=1}^{n} b_j C_{2j} \\ . \\ . \\ \sum_{j=1}^{n} b_j C_{nj} \end{bmatrix}$$
$$= \frac{1}{det(A)} \begin{bmatrix} det(B_1) \\ det(B_2) \\ . \\ . \\ det(B_n) \end{bmatrix} \tag{3.93}$$

B_i is a matrix obtained by replacing the i^{th} column of A with the vector $\mathbf{b}$ and keeping all other columns unaltered. This rule is never used in practical computations; the explicit formula only helps in analysis or derivation.

3.11 Eigenvalues and Eigenvectors

Let A be an $n \times n$ square matrix. Consider the function $f : \Re^n \rightarrow \Re^n$, defined as $f(\mathbf{x}) = A\mathbf{x}$. Suppose we are interested in vectors $\mathbf{x}$, for which, f returns a vector in the same direction as $\mathbf{x}$. Such vectors are called *eigenvectors* of A.

Eigenvector: *Vector $\mathbf{x} \in \Re^n$ is called an eigenvector of an $n \times n$ matrix A, iff*

$$A\mathbf{x} = \lambda\mathbf{x}, \ \exists \ \lambda \in \Re \tag{3.94}$$

The scalar λ is called an **eigenvalue** *of A, corresponding to the eigenvector $\mathbf{x}$.*

We will consider some special examples of eigenvectors and eigenvalues.

- For the simple case of $\lambda = 0$, any $\mathbf{x} \in N(A)$ is an eigenvalue of A. Thus, if A is singular, so that $N(A) \neq \{\}$, $\lambda = 0$ and $\mathbf{x} \in N(A)$ are a valid eigenvalue-eigenvector pair.
- If A happens to be a projection matrix, (*c.f.*, Section 3.9.1), *i.e.*,

$$A = \mathbf{s}\mathbf{s}^T \frac{1}{\mathbf{s}^T\mathbf{s}}$$

for some $\mathbf{s} \in \Re^n$. Recall that, $A\mathbf{x} = \mathbf{x}$ for any $\mathbf{x}$ in the column space of A. Therefore, $\mathbf{x} = \rho\mathbf{s}$ is an eigenvector of A for any $\rho \in \Re$, with 1 as the corresponding eigenvalue. As discussed above, any $\mathbf{x} \in N(A)$ is also an eigenvector of A with a corresponding eigenvalue of 0. However, for any other $\mathbf{x} \notin C(A)$, $A\mathbf{x} = c\mathbf{s}$ and therefore x is not an eigenvector.

Consider the permutation matrix P_{23} in (3.95):

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.95}$$

By inspection, we find that P_{23} has atleast three eigenvectors, *viz.*, $x_1 = [1 \quad 0 \quad 0]$ with eigenvalue $\lambda_1 = 1$, $x_2 = [0 \quad 1 \quad 1]$ with eigenvalue $\lambda_2 = 1$, and $x_3 = [0 \quad -1 \quad 1]$ with eigenvalue $\lambda_3 = -1$. Does P_{23} have any more eigenvectors? The answer is no. It turns out that any $n \times n$ matrix has exactly n orthonormal eigenvectors. Moreover, the trace of a matrix (*i.e.*, the sum of its diagonal entries) always equals the sum of the eigenvalues corresponding to the orthonormal eigenvectors.

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

Thus, if we knew $n-1$ eigenvalues of a matrix, we could easily determine its n^{th} eigenvalue. We will defer this discussion to a later part of this chapter.

3.11.1 Solving for Eigenvalues

The equation (3.94) defining the criterion for an eigenvalue $\mathbf{x}$ can we re-written as in (3.96).

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3.96}$$

For a solution $\mathbf{x}$ to exist, $A - \lambda I$ must be singular (*i.e.*, non-invertible) and $\mathbf{x}$ must lie in the null space $N(A - \lambda I)$. Therefore, $det(A - \lambda I) = 0$ is a necessary and sufficient condition for λ to be an eigenvalue. Once the eigenvalue λ is determined, the corresponding eigenvectors can be determined by computing $N(A - \lambda I)$, a procedure that has been already discussed in Section 3.5.2. We will therefore first discuss the procedure for computing the solution to

$$det(A - \lambda I) = 0 \tag{3.97}$$

As an example, when we apply the criterion in (3.97), to the matrix P_{23}, we get solutions as shown in (3.98):

$$\begin{aligned} det(P_{23} - \lambda I) &= (1 - \lambda)\lambda^2 = 0 \\ &\Rightarrow \lambda = 1 \; or \; \lambda = -1 \end{aligned} \tag{3.98}$$

Substituting these two values into the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$, we get one matrix for each possible value of λ. It can be verified that the basis for the null space of $(A - \lambda I)$ obtained using the elimination process discussed in Section 3.6 (particularly, equation 3.27) is indeed $[1 \;\; 0 \;\; 0]^T$ and $[0 \;\; 1 \;\; 1]^T$ for eigenvalue $\lambda_1 = 1$, and $[0 \;\; -1 \;\; 1]$ for eigenvalue $\lambda_3 = -1$.

3.11.2 Some Properties of Eigenvalues and Eigenvectors

How are the eigenvectors and eigenvalues of a matrix affected when transformations are performed on the matrix? Below, we list some properties of eigenvalues with respect to matrix transformations.

1. If $A\mathbf{x} = \lambda\mathbf{x}$, then $(A + kI)\mathbf{x} = (\lambda + k)\mathbf{x}$. That is, the eigenvalues of $A + kI$ are the eigenvalues of A, incremented by k, without any change in corresponding eigenvectors.

2. Consider the matrix R in (3.99):

$$R = \begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix} \tag{3.99}$$

   The eigenvalues of R can be found as follows: $det(R - \lambda I) = \lambda^2 + 6 = 0 \Rightarrow \lambda = \pm\sqrt{6}i$. The eigenvalues of a matrix could be complex numbers as this example illustrates. In fact, eigenvalues always appear as complex conjugates, as in this example.

3. Let λ be an eigenvalue of A and $\mathbf{x}$ its corresponding eigenvector, *i.e.*, $A\mathbf{x} = \lambda\mathbf{x}$. It can be shown that the complex conjugates $\overline{\lambda}$ and $\overline{\mathbf{x}}$ also form an eigenvalue-eigenvector pair for $\overline{A}$. Thus, $\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. If A happens to have only real entries, then, $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$.

4. The eigenvalues of upper and lower traingular matrices can be computed very easily. By derived property (4) of determinants, the determinant of an upper traingular matrix is the product of its diagonal entries. Let U be

an $n \times n$ upper traingular matrix, with u_{ij} being the entry corresponding to the i^{th} row and j^{th} column. First we note that $U - \lambda I$ will also be upper traingular, since I is upper traingular and since the sum of upper traingular matrices is also upper traingular (the space of upper traingular matrices is a vector space, as shown in Section 3.8). Now, $det(U - \lambda I) = \prod_{i=1}^{n}(u_{ii} - \lambda) = 0$. The eigenvalues correspond to solutions of this equation; they are $\lambda_i = u_{ii},\ 1 \leq i \leq n$. The eigenvectors can be computed by solving the systems $(U - \lambda_i I)\mathbf{x}_i = \mathbf{0}$ by simple back-subtitutions, as illustrated in Section 3.3.1.

5. If $\mathbf{x}$ is an eigenvector of A with a corresponding eigenvalue λ, we have $A\mathbf{x} = \lambda\mathbf{x}$. Therefore, $A^2\mathbf{x} = A(A\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Thus, $\mathbf{x}$ is an eigenvector of A^2 as well, with a corresponding eigenvalue of λ^2. This statement can be generalized: *If* $\mathbf{x}$ *is an eigenvector of* A *with a corresponding eigenvalue* λ, $\mathbf{x}$ *is also an eigenvector of* A^k, *with corresponding eigenvector* λ^k.

6. The eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of a matrix A are linearly independent if all its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are different. This can be proved by contradiction[9] However, the eigenvectors, could be independent even if eigenvalues are repeated; but it is not always true. For instance, any traingular matrix having some identical diagonal elements (as in the case of the identity matrix) has linearly independent eigenvectors, even though some eigenvalues are identical.

7. In many engineering problems, we are faced with the system of equations

$$\mathbf{b}_{i+1} = A\mathbf{b}_i,\ \forall\ i \geq 0 \tag{3.100}$$

That is, $\mathbf{b}_i = A^i\mathbf{b}_0$. If A has n linearly independent eigenvectors (so that they span $\Re^n$), these systems can be solved efficiently, by expressing $\mathbf{b}_0$ as a linear combination of the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of A.

$$\mathbf{b}_0 = \sum_{k=1}^{n} c_k\mathbf{v}_k$$

where, $c_k \in \Re,\ \forall\ 1 \leq k \leq n$. Consequently, any $\mathbf{b}_i,\ i \geq 0$ can be computed efficiently as

$$\mathbf{b}_i = \sum_{i=k}^{n} \lambda_k^i c_k\mathbf{v}_k \tag{3.101}$$

[9] Exercise.

8. Consider the fibonacci sequence $f_{i+2} = f_i + f_{i+1} i \geq 0$, with $f_0 = 0$ and $f_1 = 1$. The recurrance relation can be written as a linear system (3.102).

$$\underbrace{\begin{bmatrix} f_{i+2} \\ f_{i+1} \end{bmatrix}}_{\mathbf{b}_{i+1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} f_{i+1} \\ f_i \end{bmatrix}}_{\mathbf{b}_i} \tag{3.102}$$

Note that $\mathbf{b}_0 = [0\ 1]^T$. The system of equations (3.102) is of the same form $\mathbf{b}_{i+1} = A\mathbf{b}_i,\ \forall\ 0 \leq i \leq n$ discussed above and therefore, the expression for $\mathbf{b}_i$ can be derived using (3.101), after computing values of λ_1, $\mathbf{v}_1$, c_1 and λ_2, $\mathbf{v}_2$ and c_2. The values of $\lambda_1 = \frac{1}{2}(1 + \sqrt{5}) = 1.6180$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5}) = -0.6180$ can be computed by solving $det(A - \lambda I) = 0$. Substituting these values of λ, eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be obtained as in (3.103).

$$\mathbf{v}_1 = \begin{bmatrix} -0.8507 \\ -0.5257 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0.5257 \\ -0.8507 \end{bmatrix} \tag{3.103}$$

A closed form expression is $\mathbf{b}_i = c_1(1.6180)^i[-0.8507\ -0.525]^T - c_2(0.6180)^i[0.525\ -0.8507]^T$.

Another application of this general technique is in differential equations. Let us say we are given the differential equation $x'' + a_1 x' + a_2 x = 0$. This equation can be equivalently expressed as

$$\mathbf{y}' = \underbrace{\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}}_{A} \mathbf{y} \tag{3.104}$$

where, $\mathbf{y} = [x'\ \ x]^T$. The n^{th} derivative of x can expressed in a closed form by determining the eigenvalues and eigenvectors of A.

9. If λ is an eigenvalue of a matrix A, then it is also an eigenvalue of A^T. This is because $det(A - \lambda I) = det\left((A - \lambda I)^T\right) = det(A^T - \lambda I)$. The eigenvectors for the same eigenvalues could however differ between A and A^T.

10. Another general property of any square matrix is that the sum of its eigenvalues equals its trace. Additionally, the product of its eigenvalues equals its determinant. Consequently, for any 2×2 matrix, if the trace is negative and the determinant positive, the real parts of both its eigenvalues must be negative.

11. A Markov matrix[10] M is an $n \times n$ matrix such that (1) all its entries are ≥ 0 and (2) the sum of all entries in each column is 1. An example Markov matrix M_3 is

$$M = \begin{bmatrix} 0.1 & 0.25 & 0.3 & 0.35 \\ 0.2 & 0.25 & 0.3 & 0.05 \\ 0.3 & 0.25 & 0.4 & 0.15 \\ 0.4 & 0.25 & 0 & 0.45 \end{bmatrix} \tag{3.105}$$

A very special property of markov matrices is that exactly one eigenvalue of any markov matrix equals 1 and the rest of its eigenvalues are strictly less than 0. For example, the M_3 has following eigenvalues: $1.0000, -0.2168, 0.3428, 0.0740$. The first part of this property can be proved as follows. The matrix $M - I$ is singular, because the sum of the rows is a zero vector. Therefore, $det(M - I) = 0$. Thus, $\lambda = 1$ must be an eigenvalue of M.

In probabilistic models, we often have systems of the form $\mathbf{p}_{i+1} = A\mathbf{p}_i$, $\forall\, i \geq 0$, similar to equation (3.100). A closed form solution can be obtained using the idea of (3.101)

$$\mathbf{p}_i = \sum_{i=k}^{n} \lambda_k^i c_k \mathbf{v}_k$$

where, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the eigenvectors of M and its eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$. If $\lambda_1 = 1$, then $\lambda_i < 1, \forall\ 2 \leq i \leq n$. Hence, as $i \to \infty$, $\mathbf{p}_i \to c_1\mathbf{v}_1$.

12. If A is an $n \times n$ matrix with real valued entries and is symmetric, *i.e.*, $A = A^T$, then, its eigenvalues are real. Further, the eigenvectors of a symmetric matrix can be chosen to be orthogonal. In mathematics, this is called the *spectral theeorem* while in mechanics it is called the *principal axis theorem.*

Theorem 37 *If A is symmetric then (1) all its eigenvalues are real and (2) there exists and orthonormal basis Q of A, consisting of its eigenvectors.*

Proof for part (1): Let λ be an eigenvalue of A and $\mathbf{x}$ be its corresponding eigenvector; $A\mathbf{x} = \lambda\mathbf{x}$. Then, premultiplying both sides by $\overline{\mathbf{x}}^T$, we get

$$\overline{\mathbf{x}}^T A\mathbf{x} = \lambda \overline{\mathbf{x}}^T \mathbf{x} \tag{3.106}$$

[10]The matrix entries of a markov entries represent probabilities of transitions within/between states.

As mentioned earlier, the complex conjugates $\overline{\lambda}$ and $\overline{x}$ also form an eigenvalue-eigenvector pair for a real matrix A; $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. This implies that $\overline{\mathbf{x}}^T A^T = \overline{\mathbf{x}}^T A = \overline{\lambda}\overline{\mathbf{x}}^T$ and therefore,

$$\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} \tag{3.107}$$

We note that the left hand sides of (3.106) and (3.107) are the same. Equating the right hand sides of these equations,

$$\lambda\overline{\mathbf{x}}^T\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} \tag{3.108}$$

$\overline{\mathbf{x}}^T\mathbf{x}$ is always real and non-negative. It is 0 only if $\mathbf{x} = \mathbf{0}$. Therefore, $\lambda = \overline{\lambda} \Rightarrow \lambda \in \Re$. □

13. If A is a real symmetric matrix, the number of positive pivots and number of negative pivots are respectively equal to the number of positive and negative eigenvalues.

14. Two $n \times n$ matrices A and B are called *similar* if there exists an invertible $n \times n$ matrix M such that $M^{-1}BM = A$. A property of similar matrices is that they have same determinants, since $det(A) = det(M^{-1})det(B)det(M) = \frac{1}{det(M)}det(B)det(M) = det(B)$. A more fundamental property is that similar matrices have the same eigenvalues, though they could differ in their eigenvectors.

Theorem 38 *If A and B are similar matrices, they have the same eigenvalues.*

Proof: Let λ be an eigenvalue of A. Since A and B are similar, there exists an invertible matrix M such that, $M^{-1}BM = A$. $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow (MAM^{-1})M\mathbf{x} = \lambda M\mathbf{x} \Rightarrow B(M\mathbf{x}) = \lambda(M\mathbf{x})$, that is, if λ is an eigenvalue of A and $\mathbf{x}$ is the corresponding eigenvector, then λ is an eigenvalue of B and $M\mathbf{x}$ is its corresponding eigenvector.

Similarly, $B\mathbf{x} = \lambda\mathbf{x} \Rightarrow (M^{-1}AM)M^{-1}\mathbf{x} = \lambda M^{-1}\mathbf{x} \Rightarrow B(M^{-1}\mathbf{x}) = \lambda(M^{-1}\mathbf{x})$, that is, if λ is an eigenvalue of B and $\mathbf{x}$ is the corresponding eigenvector, then λ is an eigenvalue of A and $M^{-1}\mathbf{x}$ is its corresponding eigenvector. □

At this point, we state the observation that matrices of the form $kI_{n\times n}$ are only similar to themselves, since, for any invertible matrix M, $M^{-1}(kI_{n\times n})M = kI_{n\times n}$.

3.11.3 Matrix Factorization using Eigenvectors

Let A be an $n \times n$ matrix, with n eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_1$. Let V be a matrix with the eigenvectors as columns. Postmultiplying A by V, we get

$$AV = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \ldots \quad \lambda_n\mathbf{v}_n] = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ . & . & \ldots & . \\ . & . & \lambda_k & . \\ . & . & \ldots & . \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}}_{Eigenvalue\ matrix\ \Lambda} = V\Lambda$$

that is, $AV = V\Lambda$. The diagonal matrix Λ consists of eigenvalues along its diagonal and is called the *eigenvalue matrix*.

If the eigenvectors are linearly independent, V is invertible. Premultiplying AV by V^{-1},

$$V^{-1}AV = \Lambda$$

Another equivalent equation is

$$A = V\Lambda V^{-1} \tag{3.109}$$

This procedure of premultiplying a matrix by the inverse of its eigenvector matrix and post-multipyling it by the eigenvector matrix to obtain a diagonal matrix of its eigenvalues, is called *diagonalization*. Diagonalization can be generalized to powers of k:

$$A^k = V\Lambda^k V^{-1}$$

Thus, eigenvalues and eigenvectors provide a great way to understand the powers of a matrix. Further, if $|\lambda_i| < 1$, $\Lambda^k \to 0$, *as* $k \to \infty$. Therefore, if $|\lambda_i| < 1$, $A^k \to 0$, *as* $k \to \infty$. As another example, if we define $e^{\rho A} = \sum_{n=0}^{\infty} \frac{1}{n!}(A\rho)^n$, where $\rho \in \Re$, then using the above property, it can be shown that $e^{\rho A} = Ve^{\rho\Lambda}V^{-1} = V\ diag(e^{\rho\lambda_1}, e^{\rho\lambda_2}, \ldots, e^{\rho\lambda_n})V^{-1}$, where $diag(c_1, c_2, \ldots, c_n)$ returns an $n \times n$ diagonal matrix with the i^{th} diagonal entry as c_i.

If A is symmetric, the eigenvector matrix V could be chosen to be a matrix of orthonormal vectors, denoted by Q. Note that $Q^{-1} = Q^T$. Thus, for a symmetric A, the equation (3.109) can be re-written as:

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i(\mathbf{q}_i\mathbf{q}_i^T) \tag{3.110}$$

From Section 3.9.1, we recall that $(\mathbf{q}_i\mathbf{q}_i^T)$ is a projection matrix. Moreover, if $i \neq j$, $(\mathbf{q}_i\mathbf{q}_i^T)$ is orthogonal to $(\mathbf{q}_j\mathbf{q}_j^T)$. This gives us another perspective of symmetric matrices - as a linear combination of orthogonal projection matrices. Also, since Q is of rank 1 and invertible, we can infer that A is similar to Λ. The diagonal matrix Λ can be thought of as a canonical form for the family of matrices similar to A. However, if A is not a full rank matrix, there exists an 'almost diagonal form', called the *Jordan form* [**?**], which is similar to A, containing the eigenvalues of A along its diagonal, with the only other non-zero entries being along the super-diagonal.

One more illustration of the utility of matrix factorization using eigenvectors is the interpretation of level sets involving the quadratic form $\mathbf{x}^TA\mathbf{x} = \mathbf{x}^TQ\Lambda Q^T\mathbf{x}$ for a symmetric matrix A. The *level set* of a real-valued function f of $\mathbf{x} \in \Re^n$ is a set of the form $\{\mathbf{x}|f(\mathbf{x}) = c\}$, where c is a constant. Using the eigenvalue factorization of matrices, the level set $\{\mathbf{x}|\mathbf{x}^TQ\Lambda Q^T\mathbf{x} = c\}$ can be interpreted as an ellipsoid in n dimensions, with each eigenvector-eigenvalue pair specifying the direction and the length respectively of an axis of the ellipsoid.

3.12 Positive Definite Matrices

Positive definite matrix: *A positive definite (p.d.) matrix is a symmetric matrix with all positive eigenvalues. That M is a p.d. matrix is also denoted by $M > 0$.*

By virtue of property of symmetric matrices, all the pivots in the rref of a p.d. matrix are also positive. Since the determinant of matrix equals the product of its eigenvalues, the determinant of a p.d. matrix is also positive; however, it is not necessary that a matrix with positive determinant is also p.d.

A matrix is called *positive semi-definite* (p.s.d.), if all its eigenvalues are non-negative. That M is p.s.d. is also denoted by $M \geq 0$.

3.12.1 Equivalent Conditions

We will list down some necessary and sufficient conditions for a matrix A to be positive definite or positive semi-definite:

1. *A matrix A is p.d. iff all its n leading principal minors (c.f. Section 3.10.1) are positive.* As an example, if A is a 2×2 matrix, we must have $a_{11} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$ in order for A to be p.d. On the other hand, if all its principal minors are non-negative, the matrix is p.s.d.

2. Another equivalent definition for positive definiteness is: *A matrix A is p.d. iff,* $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^TA\mathbf{x} > 0$. This condition can be rewritten as $\forall\ \mathbf{x} \neq \mathbf{0},\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j > 0$. If $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^TA\mathbf{x} \geq 0$, A is p.s.d.

3. The condition $\forall\ \mathbf{x} \neq \mathbf{0},\ \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j > 0$ involves a quadratic expression. The expression is guaranteed to be greater than $0\ \forall\ \mathbf{x} \neq \mathbf{0}$ *iff* it can be expressed as $\sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{i-1} \beta_{ij} x_{ij} + x_{ii} \right)^2$, where $\lambda_i \geq 0$. This is possible *iff* A can be expressed as LDL^T, where, L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix of all positive diagonal entries. Or equivalently, it should be possible to factorize A as RR^T, where $R = LD^{1/2}$ is a lower traingular matrix. Note that any symmetric matrix A can be expressed as LDL^T, where L is a lower traingular matrix with 1 in each diagonal entry and D is a diagonal matrix; positive definiteness has only an additional requirement that the diagonal entries of D be positive. This gives another equivalent condition for positive definiteness: *Matrix A is p.d. if and only if, A can be uniquely factored as $A = RR^T$, where R is a lower traingular matrix with positive diagonal entries.* This factorization of a p.d. matrix is reffered to as *Cholesky factorization.*

   Recall that Guass elimination on a matrix A yields its factorization as $A = LU$ and the diagonal entries of L are pivots. Therefore, if A is symmetric matrix such that Guass elimination on it yields positive pivots, A is positive definite.

To illustrate the equivalence of the above definitions of positive definiteness, consider the matrix P below:

$$P = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 10 & 14 & 4 \\ 2 & 14 & 21 & 9 \\ 1 & 4 & 9 & 20 \end{bmatrix} \tag{3.111}$$

The matrix is positive definite and this can be proved by showing any of the following properties:

1. All the eigenvalues of P, *viz.*, $\lambda_1 = 0.1644,\ \lambda_2 = 0.9371,\ \lambda_3 = 14.4091,\ \lambda_4 = 36.4893$ are positive. and therefore $P > 0$.

2. The principal minors of P are $1, 9, 9$ and 81. All the four principal minors are positive and thus $P > 0$.

3. Matrix P can be factorized as LL^T, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 3 & 3 \end{bmatrix} \tag{3.112}$$

Since L is lower traingular and since all its diagonal entries are positive, $P > 0$.

3.12.2 Some properties

We will list some properties of positive definite matrices, using an appropriate definition of positive definiteness as required.

1. If matrices $A > 0$ and $B > 0$, then $A + B > 0$. This follows from the fact that $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^T A\mathbf{x} > 0$ and $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^T B\mathbf{x} > 0$ implies that $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^T(A+B)\mathbf{x} > 0$. Similarly, $AB > 0$ and for any $c > 0$, $cA > 0$.

2. If $A > 0$, then $\forall\ \mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^T A\mathbf{x} > 0$ implies $(\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x} > 0$, that is, $A^T > 0$.

3. Let A be an $m \times n$ matrix. Recall from Section 3.9.2, the important matrix $A^T A$ which happened to be an $n \times n$ matrix. If A is full column rank, the only vector in its null space is $\mathbf{0}$. Note that $\forall\ \mathbf{x} \neq \mathbf{0},\ \mathbf{x}^T A^T A\mathbf{x} = ||A\mathbf{x}||^2 > 0$. Thus, $A^T A$ is always p.d. if A is non-singular.

4. Every p.d. matrix is invertible and its inverse is also p.d. That is, if $A > 0$ then A^{-1} exists and $A^{-1} > 0$.

5. If $A > 0$, the diagonal entries of A are real and positive. Consequently, the trace $tr(A)$ is also positive.

Testing for positive definiteness of a matrix arises in several applications, including optimization. Determining the local minimum of a function $f(\mathbf{x}),\ \mathbf{x} \in \mathcal{D},\ \mathcal{D} \subseteq \Re^k$ involves determining points $\hat{\mathbf{x}}$ at which $\nabla f(\hat{\mathbf{x}}) = 0$ and $\nabla^2 f(\hat{\mathbf{x}}) > 0$ (positive curvature at $\hat{\mathbf{x}}$).

3.13 Singular Value Decomposition

In Section 3.11.3, we discussed that a full rank symmetric matrix can be factorized into $Q\Lambda Q^T$, where, Q is an orthonormal matrix and Λ is a diagonal matrix. This factorization can be extended to any matrix and it is called *Singular Value Decomposition*, abbreviated as *SVD*. The singular value decomposition of any $m \times n$ matrix A is factorization of A as $U\Sigma V^T$, where Σ is a diagonal matrix and U and V are orthonormal matrices.

We will contruct the matrices U and V as follows. Let r be the rank of A and let

- $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be an orthonormal basis for the column space of A.
- $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_n$ be an orthonormal basis for the null space of A.
- $\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \ldots, \mathbf{u}_m$ be an orthonormal basis for the null space of A^T.
- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be such that $\mathbf{x}_i = A^T\mathbf{u}_i$ and $\mathbf{v}_i = \frac{1}{||\mathbf{x}_i||}\mathbf{x}_i$.

The relationship between $\mathbf{u}_i$ and $\mathbf{v}_i$ is therefore $A^T\mathbf{u}_i = \sigma_{ii}\mathbf{v}_i$, with

$$\sigma_{ii} = \begin{cases} ||A^T\mathbf{u}_i|| & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases} \tag{3.113}$$

This system of equations can written in matrix form as

$$A^T U = V\Sigma \tag{3.114}$$

where, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are the columns of U and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the columns of V. Σ is an $n \times n$ diagonal matrix with its ij^{th} entry given by σ_{ij}, such that

$$\sigma_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ ||A^T\mathbf{u}_i|| & \text{if } i = j \text{ and } i \leq r \\ 0 & \text{if } i = j \text{ and } i > r \end{cases} \tag{3.115}$$

It can be shown that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are orthonormal and form a basis for the row space of A. Theorem 34 stated that the row space $C(A^T)$ and right null space $N(A)$ are orthogonal complements. Similarly, the column space $C(A)$ and left null space $N(A^T)$ are orthogonal complements. Therefore, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is an orthonormal basis for $\Re^m$, while $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthonormal basis for $\Re^n$.

Since $U^{-1} = U^T$, we can rewrite (3.114) as

$$A = U\Sigma V^T \tag{3.116}$$

Furthermore, $AA^T = U\Sigma^2 U^T$ and $A^T A = V\Sigma^2 V^T$, which are spectral decompositions, implying that the columns of U and V are eigenvectors of AA^T and $A^T A$ respectively and the diagonal entries of Σ are square roots of the eigenvalues of AA^T (or equivalently $A^T A$).

As an example, if P is the full rank, symmetric matrix in (3.111), the matrices U, Σ and V are

$$U = \begin{bmatrix} -165/2423 & 76/4167 & 637/688 & -892/2403 \\ -467/1012 & 373/992 & -577/1726 & -757/1036 \\ -367/508 & 48/133 & 318/1909 & 869/1536 \\ -172/337 & -407/477 & -329/5765 & -211/2328 \end{bmatrix} \tag{3.117}$$

$$\Sigma = \begin{bmatrix} 1715/47 & 0 & 0 & 0 \\ 0 & 11657/809 & 0 & 0 \\ 0 & 0 & 477/509 & 0 \\ 0 & 0 & 0 & 265/1612 \end{bmatrix} \tag{3.118}$$

$$V = \begin{bmatrix} -165/2423 & 76/4167 & 637/688 & -892/2403 \\ -467/1012 & 373/992 & -577/1726 & -757/1036 \\ -367/508 & 48/133 & 318/1909 & 869/1536 \\ -172/337 & -407/477 & -329/5765 & -211/2328 \end{bmatrix} \tag{3.119}$$

On the other hand, if P is a singular matrix of rank 2, given by

$$P = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 2 \end{bmatrix} \tag{3.120}$$

then P can be decomposed into the following matrices:

$$U = \begin{bmatrix} -1301/3398 & 794/1101 & -780/1351 \\ -450/1039 & -715/1033 & -780/1351 \\ -337/413 & 203/6999 & 780/1351 \end{bmatrix} \tag{3.121}$$

$$\Sigma = \begin{bmatrix} 2565/299 & 0 & 0 \\ 0 & 687/1076 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.122}$$

$$V = \begin{bmatrix} -799/1854 & -647/717 & 0 \\ -1814/2119 & 453/1108 & -228/721 \\ -567/1987 & 151/1108 & 684/721 \end{bmatrix} \tag{3.123}$$

Notice that, since P is singular and of rank 2, its null space has dimension 1 and one of its eigenvalues is 0.

3.13.1 Pseudoinverse

The SVD of a matrix that is not full rank (such as P in (3.120)) can be used to compute its so-called *Moore-Penrose pseudoinverse*.

Pseudoinverse: *The pseudoinverse A^+ of an $m \times n$ matrix A is a unique $n \times m$ matrix, satisfying all the following criteria:*

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $\overline{(AA^+)}^T = AA^+$
4. $\overline{(A^+A)}^T = A^+A$

The pseudoinverse of a non-singular square matrix is the same as its inverse. A pseudoinverse of a rectangular matrix of full column rank is the left inverse, while a pseudoinverse of a rectangular matrix of full row rank is the right inverse (*c.f.* Section 3.4.2).

Consider an $n \times n$ diagonal matrix Σ having rank k.

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_{22} & \dots & 0 & 0 & \dots & 0 \\ . & . & \dots & . & . & \dots & . \\ . & . & \dots & . & . & \dots & . \\ 0 & 0 & \dots & \sigma_{kk} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ . & . & \dots & . & . & \dots & . \\ . & . & \dots & . & . & \dots & . \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.124}$$

The pseudoinverse Σ^+ of Σ is:

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_{11}} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_{22}} & \dots & 0 & 0 & \dots & 0 \\ . & . & \dots & . & . & \dots & . \\ . & . & \dots & . & . & \dots & . \\ 0 & 0 & \dots & \frac{1}{\sigma_{kk}} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ . & . & \dots & . & . & \dots & . \\ . & . & \dots & . & . & \dots & . \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.125}$$

The pseudoinverse P^+ of any non full rank matrix P can be computed using its singular value decomposition $U\Sigma V^T$ and the pseudoinverse Σ^+ of the diagonal matrix Σ as:

$$P^+ = V\Sigma^+ U \tag{3.126}$$