SHT: Separating hyperplane theorem (a fundamental theorem)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq \mathbf{b}$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

SHT: Separating hyperplane theorem (restated)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
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That is, the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Proof of the Separating Hyperplane Theorem

We first note that the set $\mathcal{S} = \left\{\mathbf{x} - \mathbf{y} | \mathbf{x} \in \mathcal{C}, \mathbf{y} \in \mathcal{D}\right\}$ is convex, since it is the sum of two convex sets. Since $\mathcal{C}$ and $\mathcal{D}$ are disjoint, $\mathbf{0} \notin \mathcal{S}$. Consider two cases:

1. Suppose $\mathbf{0} \notin \mathit{closure}(\mathcal{S})$. Let $\mathcal{E} = \{0\}$ and $\mathcal{F} = \mathit{closure}(\mathbf{S})$. Then, the euclidean distance between $\mathcal{E}$ and $\mathcal{F}$, defined as
   $\mathit{dist}(\mathcal{E};\mathcal{F}) = \mathit{inf}\left\{||\mathbf{u} - \mathbf{v}||_2 | \mathbf{u} \in \mathcal{E}, \mathbf{v} \in \mathcal{F}\right\}$
   is positive, and there exists a point $\mathbf{f} \in \mathcal{F}$ that achieves the minimum distance, i.e., $||\mathbf{f}||_2 = \mathit{dist}(\mathcal{E},\mathcal{F})$. Define $\mathbf{a} = \mathbf{f}$, $\mathbf{b} = ||\mathbf{f}||_2$. Then $\mathbf{a} \neq \mathbf{0}$ and the affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b = \mathbf{f}^T(\mathbf{x} - \frac{1}{2}\mathbf{f})$ is nonpositive on $\mathcal{E}$ and nonnegative on $\mathcal{F}$, *i.e.*, that the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{E}$ and $\mathcal{F}$. Thus, $\mathbf{a}^T(\mathbf{x} - \mathbf{y}) > 0$ for all $\mathbf{x} - \mathbf{y} \in \mathcal{S} \subseteq \mathit{closure}(\mathcal{S})$, which implies that, $\mathbf{a}^T\mathbf{x} \geq \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and $\mathbf{y} \in \mathcal{D}$.

Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in \textit{closure}(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has empty interior, it must lie in an affine set of dimension less than n, and any hyperplane containing that affine set contains $\mathcal{S}$ and is a hyperplane. In other words, $\mathcal{S}$ is contained in a hyperplane $\{\mathbf{z}|\mathbf{a}^T\mathbf{z} = b\}$, which must include the origin and therefore $b = 0$. In other words, $\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and all $\mathbf{y} \in \mathcal{D}$ gives us a trivial separating hyperplane.

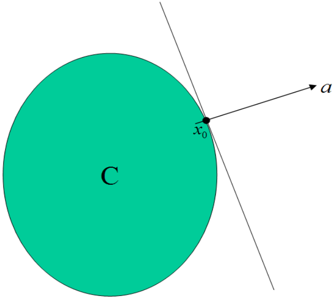
Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in closure(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has a nonempty interior, consider the set
     $\mathcal{S}_{-\epsilon} = \left\{\mathbf{z} | B(\mathbf{z}, \epsilon) \subseteq \mathcal{S}\right\}$
     where $B(\mathbf{z}, \epsilon)$ is the Euclidean ball with center $\mathbf{z}$ and radius $\epsilon > 0$. $\mathcal{S}_{-\epsilon}$ is the set $\mathcal{S}$, shrunk by ϵ. *closure* $(\mathcal{S}_{-\epsilon})$ is closed and convex, and does not contain $\mathbf{0}$, so as argued before, it is separated from $\{\mathbf{0}\}$ by atleast one hyperplane with normal vector $\mathbf{a}(\epsilon)$ such that
     $\mathbf{a}(\epsilon)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{\epsilon}$
     Without loss of generality assume $||\mathbf{a}(\epsilon)||_2 = 1$. Let ϵ_k, for $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim\limits_{k \to \infty} \epsilon_k = 0$. Since $||\mathbf{a}(\epsilon_k)||_2 = 1$ for all k, the sequence $\mathbf{a}(\epsilon_k)$ contains a convergent subsequence, and let $\bar{\mathbf{a}}$ be its limit. We have
     $\mathbf{a}(\epsilon_k)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{-\epsilon_k}$
     and therefore $\bar{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in$ *interior*$(\mathcal{S})$, and $\bar{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}$, which means
     $\bar{\mathbf{a}}^T\mathbf{x} \geq \bar{\mathbf{a}}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$, and $\mathbf{y} \in \mathcal{D}$.

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Positive Semidefinite Cone and Convex Analysis

More on Convex Sets and Advanced Material on Convex Analysis

- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Linear program and dual of LP.
- Properties of dual cones.
- Conic Program.
- Generalized Inequalities.

Positive semidefinite cone: Notes

1. Claim : $(S_+^n)^* = (S_+^n)$
2. i.e. $<$X,Y$>$ = tr(X^TY) = tr(XY) $\geq 0 \; \forall$ X $\in (S_+^n)$ iff Y $\in (S_+^n)$

Proof:

1.
    1. Let us say Y $\notin S_+^n$. That is $\exists$ z $\in \Re^n$ s.t. $z^T Yz =$ tr$(zz^T Y) < 0$
    2. i.e. $\exists$ X = $zz^T \in S_+^n$ s.t. $<$X,Y$>$ < 0
    3. $\implies$ Y $\notin (S_+^n)^*$
2.
    1. Suppose Y,X $\in S_+^n$. Any X $\in S_+^n$ can be written in terms of eignvalue decomposition as:
    2. X = $\sum_{i=1:n} \lambda_i u_i u_i^T$ $(\lambda_i \geq 0)$
    3. $\therefore$ $<$Y,X$>$ = tr(YX) = tr(Y $\sum_{i=1:n} \lambda_i u_i u_i^T$) = $\sum_{i=1:n} \lambda_i tr(Yu_i u_i^T) = \sum_{i=1:n} \lambda_i u_i^T Y u_i \geq 0$.
    4. Since $(\lambda_i \geq 0)$ and $(u_i^T Y u_i \geq 0$ as Y $\in S_+^n)$
    5. $\implies$ Y $\in (S_+^n)^*$

Positive semidefinite cone: Questions

1. Q) Is there some connection between $Y = yy^T$ used for $S^n_+ = \{X \in S^n \mid <yy^T, X> \geq 0\}$ and $(S^n_+)^* = (S^n_+)$.
   - (To be revisited as H/W)
2. Q) $(S^n_{++})^* = ?$, $\text{int}(S^n_+) = (S^n_{++})$
   - Ans: $(S^n_{++})^* = (S^n_+)$, (will be done formally for general case of convex cones)
   - C = convex cone, $C^{**} = \text{cl}(C)$
3. Q) Consider an application of psd cone for optimization. (thru LP)
   1. We will first see (weak) duality in a linear optimization problem (LP).
   2. Next we look at generalized (conic) inequalities and the properties that the cone must satisfy for the inequality to be a valid inequality.
   3. Next, we generalize LP to conic program (CP) using generalized inequality and realize weak duality for CP thru dual cones.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x} + b) \leq 0$
   - $\implies c^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x} + b)$
   - $\implies c^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $c^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,
   $$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T\mathbf{b}, & \text{if } A^T\lambda = c \\ -\infty, & \text{otherwise} \end{cases}$$
   - Note: LHS $(\mathbf{c}^T\mathbf{x})$ is independent of λ and R.H.S $(\lambda^T\mathbf{b})$ is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded) $\geq$ Dual LP (upper bounded):
   $(\min_{\mathbf{x}\in\Re^n} \mathbf{c}^T\mathbf{x}$, s.t. $A\mathbf{x} \geq \mathbf{b}) \geq (max_{\lambda\geq 0}\mathbf{b}^T\lambda$, s.t. $A^T\lambda = \mathbf{c})$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Note: $-A\mathbf{x} + b \leq 0$ can be rewritten as $A\mathbf{x} \geq 0$.
   - So, constraint is $A\mathbf{x} - b \in R^n_+$
   - Note: R^n_+ is a CONE. How about defining generalized inequality for a cone K as: $c \geq_K d$ iff $c - d \in K$
2. So, a generalized conic program can be defined as:
   $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
   - That is, constraint is $A\mathbf{x} - b \in K$.

Properties of dual cones

1. If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone.
   - We have already proven that C^* is a closed convex cone.
   - C^* = intersection of infinite topological half spaces.
   - $C^* = \cap_{\mathbf{x} \in C} \{y | y \in X, < \mathbf{y}, \mathbf{x} > \geq 0\}$
   - $\implies C^*$ is closed.
2. $C_1 \subseteq C_2 \implies C_2^* \subseteq C_1^*$.
3. interior$(C^*) = \{\mathbf{y} \in X | < \mathbf{y}, \mathbf{x} >> 0\}$
4. If C is cone and has $int(C) \neq \emptyset$ then C^* is pointed.
   - Since; if $\mathbf{y} \in C^*$ & $-\mathbf{y} \in C^*$, then $\mathbf{y} = 0$.
5. If C is cone then closure$(C) = C^{**}$
   - If $C =$ open half space, then $C^{**} =$ closed half space.
6. If closure of C is pointed, then interior$(C^*) \neq \phi$.

S is called conically spanning set of cone K iff $conic(S) = K$.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)
  - i.e. K has no straight lines passing through O.
  - i.e. if $-a, a \in K$, then a $= 0$

examples

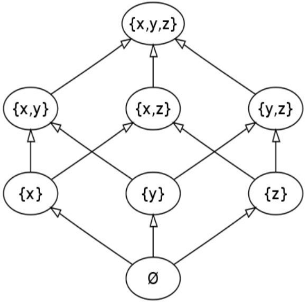
- non-negative orthant $K = R^n_+ = \{\mathbf{x} \in \Re^n | \mathbf{x}_i \geq 0, i = 1, ..., n\}$
- positive semidefinite cone $K = S^n_+$
- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

1. Reflexivity: $a \geq a$;
2. Anti-symmetry: if both $a \geq b$ and $b \geq a$, then $a = b$;
3. Transitivity: if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. Compatibility with linear operations:
    1. Homogeneity: If $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$, i.e. one can multiply both sides of an inequaility by a nonnegative real.
    2. Addititvity: if both $a \geq b$ abd $c \geq d$, then $a + c \geq b + d$, i.e. One can add two inequalities of the same sign.

Example of Partial Order

- Example of Partial Order $\subseteq$ over sets
- The Hasse diagram of the set of all subsets of a three-element set $\{\mathbf{x}, y, z\}$, ordered by inclusion(Inclusion, i.e. the Partial Order $\subseteq$):
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Dual Cones and Generalized Inequalities

Instructor: Prof. Ganesh Ramakrishnan

Contents: Vector Spaces beyond $\Re^n$

- Recap: Linear program (LP) & dual of LP.
- Recap: Conic program.
- Recap: Linear program (LP) & dual of LP.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+\mathbf{b}\leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in \Re^n_+$)
   - Then $\lambda^T(-A\mathbf{x}+\mathbf{b}) \leq 0$
   - $\implies \mathbf{c}^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x}+b)$
   - $\implies \mathbf{c}^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $\mathbf{c}^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T\mathbf{b} + (\mathbf{c} - A^T\lambda)^T\mathbf{x}$
   - Thus,
   $$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T b, & \text{if } A^T\lambda = \mathbf{c} \\ -\infty, & \text{otherwise} \end{cases}$$
   - Note: LHS ($c^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T b$) is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\geq$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x}\in\Re^n} \mathbf{c}^T\mathbf{x}, s.t. A\mathbf{x} \geq b) \geq (max_{\lambda\geq 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

1. A generalized conic program can be defined as:
   $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
   - That is, constraint is $A\mathbf{x} - b \in K$.
2. Q: Has to generalize $-A\mathbf{x} + b \leq 0$ to $-A\mathbf{x} + b \leq_K 0$ s.t. $\leq_K$ is a generalized inequality & K some set?
3. What properties should K satisfy so that $\leq_K$ satisfies properties of generalized inequalities?

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

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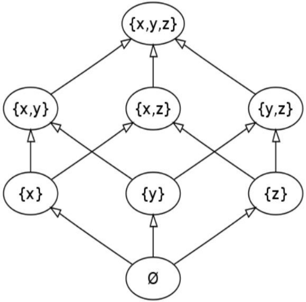
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.

Proof:

1. K being pointed convex cone $\implies$ $\geq_K$ is a partial order
    1. Reflexivity: $a \geq_K a$, since $a - a = 0 \in K$ ($\because$ K is cone)
    2. Anti-symmetry: If a $\geq_K$ b & b $\geq_K$ a then a $=$ b, since a - b $\in$ K & b -a $\in$ K $\implies$ a - b $= 0$ ($\because$ K is pointed)
    3. Transitivity: If both a $\geq_K$ b & b $\geq_K$ c then a $\geq_K$ c, since a - b $\in$ K & b -c $\in$ K $\implies$ (a - b) $+$ (b - c) $\in$ K ($\because$ K is a convex cone i.e. contain all conic combinations of points in the set)
    4. Homogeneity: If both a $\geq_K$ b & $\lambda \geq 0$ then λa $\geq_K$ λb, since a - b $\in$ K & $\lambda \geq 0$ $\implies$ λ(a - b) $\in$ K ($\because$ K is a cone)
    5. Additivity: If a $\geq_K$ b & c $\geq_K$ d then a $+$ c $\geq_K$ b $+$ d, since a - b $\in$ K & c -d $\in$ K $\implies$ (a $+$ c) - (b $+$ d) $\in$ K ($\because$ K is a convex cone)
2. $\geq_K$ is a partial order $\implies$ K being pointed convex cone

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.

Proof:

1. $\geq_K$ is a partial order $\implies$ K being pointed convex cone
   1. K is convex cone: If $\mathbf{x}, \mathbf{y} \in K$ then $\theta_1\mathbf{x} + \theta_2 y \in K \forall\ \theta_1, \theta_2 \geq 0$, since $\mathbf{x} \geq_K 0$ & $y \geq_K 0 \implies \theta_1\mathbf{x} \geq_K 0$ & $\theta_2 y \geq_K 0\ \forall\ \theta_1, \theta_2 \geq 0$ (Homogeneity of $\geq_K$) and thus $\theta_1\mathbf{x} + \theta_2 y \geq 0$ (Additivity of $\geq_K$)
   2. K is pointed: If $\mathbf{x} \in K$ & $-\mathbf{x} \in K$ then $\mathbf{x} = 0$, since $\mathbf{x} \geq_K \mathbf{x}$ & $-\mathbf{x} \geq_K 0 \implies 0 \geq_K \mathbf{x}$ (reflectivity $\mathbf{x} \geq_K \mathbf{x}$, and adding $\mathbf{x} \geq_K \mathbf{x}\&{-\mathbf{x}} \geq_K 0$ by additivity) and $-\mathbf{x} \geq_K \mathbf{x}$ (additivity on $-\mathbf{x} \geq_K 0$ & $0 \geq_K \mathbf{x}$) and similarly $\mathbf{x} \geq_K -\mathbf{x}$, and by applying anti-symmetry on $-\mathbf{x} \geq_K \mathbf{x}$ & $\mathbf{x} \geq_K -\mathbf{x}$ we get $\mathbf{x} = -\mathbf{x}$ i.e. $\mathbf{x} = 0$.

Additional properties over & above K being pointed convex cone

1. Que: Suppose $a^i \geq_K b^i$ $\forall$ i & $a^i \rightarrow a$ & $b^i \rightarrow b$, then for $a \geq_K b$ what more is required of K?
2. Ans: Necessary condition is that a^i - $b^i \rightarrow a - b \in$ K. i.e. K is closed(Also happens to be a sufficient condition).
3. Que: What is required so that $\exists$ $a >_K b$ (i.e. $b \ngeq_K a$)?
4. Ans: Sufficient condition is that $a - b \in$ int(K) i.e. int(K) $\neq \phi$ OR K has non-empty interior.

Linear program (LP) & Conic program.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+b \leq 0$

$-A\mathbf{x}+b \leq 0$ can be rewritten as $A\mathbf{x} \geq b$ or $A\mathbf{x}-b \in \Re^n_+$ Note: $\Re^n_+$ is a CONE. How about defining generalized inequality for a cone C as c $>_K$ d iff $c-d \in K$ and a generl conic program as:

1. $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x}+b \leq_K 0$

- That is, constraint is $A\mathbf{x}-b \in K$.
- K is a proper cone.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)
  - i.e. K has no straight lines passing through O.
  - i.e. if $-a, a \in K$, then a $= 0$

examples

- non-negative orthant $K = R^n_+ = \{\mathbf{x} \in \Re^n | x_i \geq 0, i = 1, ..., n\}$
- psitive semidefinite cone $K = S^n_+$
- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0$ for $t \in [0, 1]\}$
- Que: What if $n \rightarrow \infty$, can you get proper cones under additional constraints?

Linear program & its dual To Conic program and its dual.

Consider LP and its dual:

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x} + b) \leq 0$
   - $\implies c^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x} + b)$
   - $\implies c^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $c^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,

   $$c^T\mathbf{x} \geq \begin{cases} \lambda^T b, & \text{if } A^T\lambda = c \\ -\infty, & \text{otherwise} \end{cases}$$

   - Note: LHS $(\mathbf{c}^T\mathbf{x})$ is independent of λ and R.H.S $(\lambda^T\mathbf{b})$ is independent of $\mathbf{x}$.

2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\geq$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}, s.t. A\mathbf{x} \geq b) \geq (max_{\lambda\geq 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

Refer page 5 of `http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf`:

1. Conic program:
   $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
2. Generalized conic program:
   $\min_{\mathbf{x}\in V} < c, \mathbf{x} >_V$
   subjected to $A\mathbf{x} - b \in K$
3. K is a regular/proper cone.
4. We need an equivalent $\lambda \in D \supseteq K^*$ s.t.
   $<\lambda, A\mathbf{x} - b >\geq 0$.
5. This K^* s.t.
   $D = \{\lambda| < \lambda, A\mathbf{x} - \mathbf{b} >\geq 0, \ \lambda \in V \ \forall \ A\mathbf{x} - b \in K\}$
   & $D \supseteq K^*$ is dual cone of K

Dual of Conic program

1. Refer page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf:
$K^* = \{\lambda : \lambda^T \xi \geq 0 \ \forall \xi \in K\}$ is the cone dual to K.
2. With this follows weak duality theorem for CONIC PROGRAM:
Primal CP (lower bounded by dual) $\geq$ Dual CP (upper bounded by primal):
$(\min_{\mathbf{x} \in V} < c, \mathbf{x} >_V$, s.t. $< \lambda, A\mathbf{x} - \mathbf{b} > \geq 0.) \geq (max_{\lambda \in K^*} < b, \lambda >, s.t. A^T \lambda = c)$

Notes: LP and CP

1. Both LP and CP dealt with affine objectives.
2. CP dealt with the generalized conic inequalities.
3. Later, in convex optimization, we will deal with the more general convex functions in the objective.

Some Generalizations:

1. If $K = R_+^n$, the CP is an LP.
2. If $K = S_+^n$ (Set of all nXn SPD matrices), the CP is an SDP (Semi-definite program).
3. Any generic convex program can be expressed as a cone program (CP).

Dual of dual

1. If K is a closed convex cone then $K^{**} = K$.
2. More generally, if K is just a convex cone, K^{**} = closure(K) (abbreviated as Cl(K))

We will prove that if K is closed, then $K^{**} = K$:

1. $K \subseteq K^{**}$, since $\mathbf{x} \in K \implies <\mathbf{x}, \mathbf{y}> \geq 0 \ \forall \ \mathbf{y} \in K^* \implies \mathbf{x} \in K^{**}$.
2. $K^{**} \subseteq K$, we will prove by contradiction. Suppose $\mathbf{x} \in K^{**}$ but $\mathbf{x} \notin K$:
   1. K^{**} is closed since any dual cone is intersection of half spaces that are closed.
   2. $\{\mathbf{x}\}$ is a singleton set.
   3. $\implies$ by "strict hyperplane theorem" (on next page and proved later): $\exists \mathbf{a} \in V \ \& \ \mathbf{b} \in \Re$ s.t. $<\mathbf{a}, \mathbf{x}> < \mathbf{b} \& <\mathbf{a}, \mathbf{y}> \geq b \forall \ \mathbf{y} \in K$.
   4. $\implies <\mathbf{a}, \mathbf{x}> < 0 \leq <\mathbf{a}, \mathbf{y}> \forall \mathbf{y} \in K$. (Since $\mathbf{y} = 0 \in K^{**}$, Claim: $\mathbf{b} = 0$ if V is a closed convex cone)
   5. $\implies \mathbf{a} \in K^* \ \& \ \mathbf{x} \notin K^{**}$ [contradiction]

Separating hyperplane theorem (a fundamental theorem)

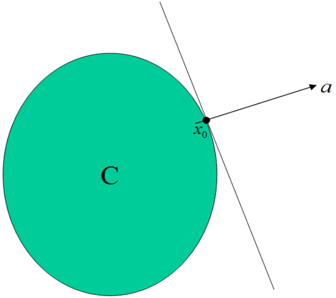
If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$, with a $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq \mathbf{b}$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq \mathbf{b}$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Dual cones and generalized inequalities

In-fact, if K is a proper cone then K^* is also proper.
$K^* = \{\lambda : \lambda^T \xi \geq 0, \ \forall \xi \in K\}$ is the cone dual to K.
Examples:

- Self-dual cones
  - $K = \Re^n_+$: $K^* = \Re^n_+$
  - $K = S^n_+$: $K^* = S^n_+$
  - $K = \{(\mathbf{x}, t) | \|\mathbf{x}\|_2 \leq t\}$: $K^* = \{(\mathbf{x}, t) | \|\mathbf{x}\|_2 \leq t\}$
- $K = \{(\mathbf{x}, t) | \|\mathbf{x}\|_1 \leq t\}$: $K^* = \{(\mathbf{x}, t) | \|\mathbf{x}\|_\infty \leq t\}$

Dual cones of proper cones are proper, hence define generalized inequalities:
$\mathbf{y} \succeq_{K^*} 0 \iff \mathbf{y}^T \mathbf{x} \geq 0$ for all $\mathbf{x} \succeq_K 0$

Minimum and minimal elements via dual inequalities

minimum element w.r.t $\preceq_K$:

- $\mathbf{x}$ is minimum element of S iff for all $\lambda \succ_{K^*} 0$, $\mathbf{x}$ is unique minimizer of $\lambda^T\mathbf{z}$ over S.

minimal element w.r.t $\preceq_K$:

- If $\mathbf{x}$ minimizes $\lambda^T\mathbf{z}$ over S for some $\lambda \succ_{K^*} 0$ then $\mathbf{x}$ is minimal
- If $\mathbf{x}$ is minimal element of convex set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that $\mathbf{x}$ minimizes $\lambda^T\mathbf{z}$ over S

From Dual of Norm Cone to Dual Norm

Let $\|.\|$ be a norm on $\Re^n$ The dual of $K = \{(\mathbf{x}, t) \in \Re^{n+1} | \|\mathbf{x}\| \leq t\}$ is:
$K^* = \{(u, v) | \in \Re^{n+1} | \|u\|_* \leq v\}$
where $\|u\|_* = sup\{u^T\mathbf{x} | \|\mathbf{x}\| \leq 1\}$
Proof: We need to show that
$\mathbf{x}^T u + tv \geq 0$ whenever $\|\mathbf{x}\| \leq t \iff \|u\|_* \leq v$