HW1: Construct a Topological space that does not have metric

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Consider $X = \{0, 1\}$ and $\mathcal{N} = \{\emptyset, \{0\}, \{0, 1\}\}$, N is a proper subset of the poweset
Consider some metric $d(.,.)$ which is 0 if both its arguments are the same and 1 otherwise. If d would be such a metric, a neighborhood (ball) of radius 0.5 around 1, that is $B(1, 0.5)$ would equal $\{1\}$, which should have been open. However, $\{1\} \notin \mathcal{N}$. Contradiction!

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Consider some metric $d(.,.)$ which is 0 if both its arguments are the same and 1 otherwise. If d would be such a metric, a neighborhood (ball) of radius 0.5 around 1, that is $B(1, 0.5)$ would equal $\{1\}$, which should have been open. However, $\{1\} \notin \mathcal{N}$. Contradiction! This was a not a proof. To prove formally, you need to show that any metric will have associated open sets that will not belong to this chosen Topology

HW2: Construct a metric space that does not have norm

Consider (again) the **discrete** metric $d(.,.)$ over a vector space V. We define $d(.,.)$ to be 0 if both its arguments are the same and 1 otherwise. While one can verify that this metric satisifies the triangle inequality. What one requires from an equivalent norm $\|.\|_n$ is that

HW2: Construct a metric space that does not have norm

Consider (again) the **discrete** metric $d(.,.)$ over a vector space V. We define $d(.,.)$ to be 0 if both its arguments are the same and 1 otherwise. While one can verify that this metric satisifies the triangle inequality. What one requires from an equivalent norm $\|.\|_n$ is that for any $\mathbf{x}, \mathbf{y} \in V$, with $\mathbf{x} \neq \mathbf{y}$, for any scalar $\alpha \neq 0$, we must have $\|\alpha\mathbf{x} - \alpha\mathbf{y}\|_n = \alpha\|\mathbf{x} - \mathbf{y}\|_n$. This measure using the norm can clearly not correspond to the **discrete** distance metric.

Every Inner product space is a normed vector space: Optional Elaborate Proof

By conjugate symmetry, we have $<\mathbf{x},\mathbf{x}>=\overline{<\mathbf{x},\mathbf{x}>}$. So $<\mathbf{x},\mathbf{x}>$ must be real.
So, we can define $\|\mathbf{x}\| = \sqrt{<\mathbf{x},\mathbf{x}>}$.
We need to prove that $\|\mathbf{x}\|$ is a valid norm:-

1. By positive definiteness: $<\mathbf{x},\mathbf{x}>\geq 0$, with equality iff $\mathbf{x}=0$. So $\|\mathbf{x}\|\geq 0$ $(=$ iff $\mathbf{x}=0)$.
2. For any complex t, $\|t\mathbf{x}\| = \sqrt{<t\mathbf{x},t\mathbf{x}>} = \sqrt{t*\overline{t}<\mathbf{x},\mathbf{x}>} = |t|\sqrt{<\mathbf{x},\mathbf{x}>}$ (as $|t| = \sqrt{t*\overline{t}}$) So $\|t\mathbf{x}\| == |t|\|\mathbf{x}\|$
3. $\|\mathbf{x}_1+\mathbf{x}_2\| = \sqrt{<\mathbf{x}_1+\mathbf{x}_2,\mathbf{x}_1+\mathbf{x}_2>} =$ $\sqrt{<\mathbf{x}_1,\mathbf{x}_1>+<\mathbf{x}_2,\mathbf{x}_2>+<\mathbf{x}_1,\mathbf{x}_2>+<\mathbf{x}_2,\mathbf{x}_1>}$ $\leq \sqrt{<\mathbf{x}_1,\mathbf{x}_1>+<\mathbf{x}_2,\mathbf{x}_2>+2\sqrt{<\mathbf{x}_1,\mathbf{x}_1><\mathbf{x}_2,\mathbf{x}_2>}}$ (**by Cauchy Schwartz inequality**)
$= \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$

Cauchy Shwarz Inequality: $| < \mathbf{u}, \mathbf{v} > | \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$

Proof:

- If $\mathbf{u} = 0$ or $\mathbf{v} = 0$, then L.H.S. $=$ R.H.S $= 0$. Hence the equality holds.
- Assume $\mathbf{u}, \mathbf{v} \neq 0$. Let $\mathbf{z} = \mathbf{u} - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}$. the projection of u on v measured along v
- By linearity of inner product in first argument, we have:
  $< \mathbf{z}, \mathbf{v} > = < \mathbf{u} - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}, \mathbf{v} > = < \mathbf{u}, \mathbf{v} > - \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>} < \mathbf{v}, \mathbf{v} > = 0$ z is orthogonal to v
- Therefore, $< \mathbf{u}, \mathbf{u} > = < \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v}, \mathbf{z} + \frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>}\mathbf{v} > = < \mathbf{z}, \mathbf{z} > + (\frac{<\mathbf{u},\mathbf{v}>}{<\mathbf{v},\mathbf{v}>})^2 < \mathbf{v}, \mathbf{v} > + 0$
- So $< \mathbf{u}, \mathbf{u} > \geq \frac{|<\mathbf{u},\mathbf{v}>|^2}{<\mathbf{v},\mathbf{v}>}$

Since <z,z> >= 0

HW3: Example of normed vector space that is not an inner product space.

infinity norm: |x| = max over i of |x_i|

HW3: Example of normed vector space that is not an inner product space.

$\|\mathbf{x}\|_p = \left[\sum_{i=1}^{n} |x_i|^p \right]^{\frac{1}{p}}$ for $p \neq 2$. For example, p=infinty gives infinity norm

HW3: Example of normed vector space that is not an inner product space.

$\|\mathbf{x}\|_p = [\ \sum_{i=1}^{n} |x_i|^p\ ]^{\frac{1}{p}}$ for $p \neq 2$.

- If $p = 2$, we get the (Eucledian) dot product:
  $\|\mathbf{x}\|_2 = [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} = [\sum_{i=1}^{n} x_i x_i]^{\frac{1}{2}} = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{<\mathbf{x},\mathbf{x}>_E}$
- Further, any inner product over a finite dimensional space over $\Re^n$ (or even $\mathbb{C}$) can be proved to have a representation in terms of the Eucledian dot product $<\mathbf{x},\mathbf{y}>_E$

HW3: Speciality of Eucledian inner product $< \mathbf{x}, \mathbf{y} >_E$ (and of $\|\mathbf{x}\|_2$) in $\Re^n$

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares). H/w
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

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- This inner product is certainly different from the conventional (Eucledian) dot product $<\mathbf{x},\mathbf{y}>_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $<\mathbf{x},\mathbf{y}>$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, <.,.>)$, the inner product $<.,.>$ (including the Eucledian one) can be represented using the basis $\mathbf{e}_1..\mathbf{e}_i..\mathbf{e}_n$ as:

$<\mathbf{u},\mathbf{v}> = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j <\mathbf{e}_i,\mathbf{e}_j> = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^TE\mathbf{b} = <\mathbf{a}^T,\mathbf{b}>_E$ where $\mathbf{u} = \sum_{i=1}^{n} a_i\mathbf{e}_i$ and $\mathbf{v} = \sum_{i=1}^{n} b_i\mathbf{e}_i$

E is the transformation (rotation/scaling) that we invoked in definiting ellipsoid using sphere

HW3: Speciality of Eucledian inner product $<\mathbf{x},\mathbf{y}>_E$ (and of $\|\mathbf{x}\|_2$) in $\Re^n$

Proof:

- In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, <.,.>)$, the inner product $<.,.>$ (including the Eucledian one) can be represented as

$$< \mathbf{u},\mathbf{v} >= \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{a_i b_j} < \mathbf{e}_i, \mathbf{e}_j >= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^T E \mathbf{b} =< \mathbf{a}^T, \mathbf{b} >_E$$

In terms of Eucledian inner prod

  - Here, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is a basis for the inner product vector space.
  - The inner product $<.,.>_E$ is the eucledian inner product. That is, $<.,.>_E = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j$.

    The (symmetric positive definite) matrix E is defined as

E is defined in terms of the new inner product

$$\boxed{E = \begin{bmatrix} <\mathbf{e}_1,\mathbf{e}_1> & <\mathbf{e}_1,\mathbf{e}_2> & \ldots\ldots & <\mathbf{e}_1,\mathbf{e}_n> \\ . & & & \\ <\mathbf{e}_n,\mathbf{e}_1> & <\mathbf{e}_n,\mathbf{e}_2> & \ldots\ldots & <\mathbf{e}_n,\mathbf{e}_n> \end{bmatrix}} \quad (34)$$

ei's need not be orthogonal

  - Note that in $\Re^n$, any inner product vector space $(\mathcal{V}, <.,.>)$ will have a basis of size at most n.

HW3: Speciality of Eucledian inner product $< \mathbf{x}, \mathbf{y} >_E$ (and of $\|\mathbf{x}\|_2$) in $\Re^n$

Thus, any inner product $< ., . >$ in $\Re^n$ can be expressed as a Eucledian inner product $< ., . >_E$, with possible rotation using a matrix R where $E = RR^T$ is a symmetric positive definite matrix[5]

[5] Recall from slides 25 to 27 that $\mathbf{x}^P\mathbf{x}$ is a norm if P is positive definite

HW3: Speciality of Eucledian inner product $< \mathbf{x}, \mathbf{y} >_E$ (and of $\|\mathbf{x}\|_2$) in $\Re^n$

Proof:

- And here is how you can create an (orthogonal) basis B for $S, < ., . >$ where $S = \{\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_k\}$
  - $\mathbf{e}_1 = \mathbf{v}_1$
  - $\mathbf{e}_2 = \mathbf{v}_2 - \frac{<\mathbf{e}_1, \mathbf{v}_2>}{<\mathbf{e}_1, \mathbf{e}_1>} \mathbf{e}_1$.
  - $\mathbf{e}_3 = \mathbf{v}_3 - \frac{<\mathbf{e}_1, \mathbf{v}_3>}{<\mathbf{e}_1, \mathbf{e}_1>} \mathbf{e}_1 - \frac{<\mathbf{e}_2, \mathbf{v}_3>}{<\mathbf{e}_2, \mathbf{e}_2>} \mathbf{e}_2$.
  - And so on..., discarding $\mathbf{e}_i$'s that turn out to be 0

constructing basis
using an inner product

HW3: Speciality of Eucledian inner product $< \mathbf{x}, \mathbf{y} >_E$ (and of $\|\mathbf{x}\|_2$) in $\Re^n$

Proof:

- And here is how you can create an (orthogonal) basis B for $S, < .,. >$ where $S = \{\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_k\}$
  - $\mathbf{e}_1 = \mathbf{v}_1$
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  - $\mathbf{e}_3 = \mathbf{v}_3 - \frac{<\mathbf{e}_1,\mathbf{v}_3>}{<\mathbf{e}_1,\mathbf{e}_1>}\mathbf{e}_1 - \frac{<\mathbf{e}_2,\mathbf{v}_3>}{<\mathbf{e}_2,\mathbf{e}_2>}\mathbf{e}_2.$
  - And so on..., discarding $\mathbf{e}_i$'s that turn out to be 0 (implying that $\mathbf{v}_i$ was linearly dependent on the preceding vectors)
  - $\mathbf{e}_k = \mathbf{v}_k - \frac{<\mathbf{e}_1,\mathbf{v}_k>}{<\mathbf{e}_1,\mathbf{e}_1>}\mathbf{e}_1 - \frac{<\mathbf{e}_2,\mathbf{v}_k>}{<\mathbf{e}_2,\mathbf{e}_2>}\mathbf{e}_2 \ldots - \frac{<\mathbf{e}_{k-1},\mathbf{v}_k>}{<\mathbf{e}_{k-1},\mathbf{e}_{k-1}>}\mathbf{e}_{k-1}.$

  We expect not more than $m \leq n$ of the k $\mathbf{e}_i$'s to be $\neq 0$.

Compact representation of Inner Product Space

- Let the linear subspace $S \subseteq V$ be associated with an inner product $< ., . >$
- Let $B = basis(S)$ with respect to the arbitrary inner product $< ., . >$ (extending results from the eucledian inner product)
- Let $dim(V) = n$, and $dim(S) = m \leq n$.

S = span(Basis) [Primal]

- Define $S^{\perp}$; the orthogonal complement ($S^{\perp} \in V$) of S as:
  $S^{\perp} = \{v \in V \mid < v, u >= 0 \; \forall \; u \in S \}$
  This implies:-
  - Both S and $S^{\perp}$ are linear subspaces of V.
  - $S \cap S^{\perp} = \{0\}$, $dim(S) + dim(S^{\perp}) = n$
  - $(S^{\perp})^{\perp} = S$. S = complement of its complement [Dual]
  - If $B^{\perp}$ is the basis for $S^{\perp}$, then $B \cup B^{\perp}$ is the basis for V.
  - $S = \{\mathbf{v} \in V \mid < \mathbf{v}, \mathbf{u} >= 0, \; \forall \; \mathbf{u} \in B^{\perp} \}$
  - $S^{\perp} = \{\mathbf{v} \in V \mid < \mathbf{v}, \mathbf{u} >= 0 \; \forall \; \mathbf{u} \in B \}$

$dim(S^{\perp}) = k - m = r$

Dual Representation: Explained with Analogy

Primal

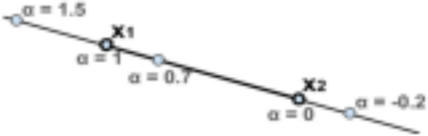
If $S \subseteq \Re^n$ and $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r\}$ is finite spanning set in $S^\perp$, then:-

- $S = (S^\perp)^\perp = \{\mathbf{x} | \mathbf{a}_i^T \mathbf{x} = 0; i = 1, ..., r\}$
- A dual representation of linear subspace S (in $\Re^n$): $\{\mathbf{x} | A\mathbf{x} = 0;\ \mathbf{a}_i^T$ is the i^{th} row of $A\}$

Dual

Affine set

- **In 2D, a line** through any two distinct points $\mathbf{x}_1, \mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.



  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ where $\alpha + \beta = 1$

- In general, A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.

Affine combination

Set A is affine iff it is closed under affine combinations of pairs of points in A

Affine set

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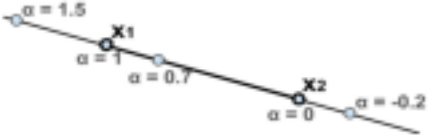
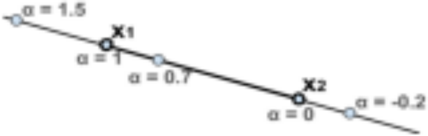


  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ where $\alpha + \beta = 1$
- In general, A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- What will $S_{\mathbf{u}} = \{\mathbf{x} - \mathbf{u} \| \mathbf{x} \in A\}$ for some fixed $\mathbf{u} \in A$ be?

Affine set

- **In 2D, a line** through any two distinct points $\mathbf{x}_1, \mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.



  $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ where $\alpha + \beta = 1$
- In general, A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta\mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- What will $S_{\mathbf{u}} = \{\mathbf{x} - \mathbf{u} \| \mathbf{x} \in A\}$ for some fixed $\mathbf{u} \in A$ be? **Ans: Vector sub-space!**
- Thus, A is affine iff for some vector sub-space S, $A (= S$ shifted by $\mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u}$ is fixed and $\mathbf{v} \in S\ \}$.

Homework (prove)

(conversely every affine set can be expressed as solution set of system of linear equations)

Affine sets: Dual Description

- **Dual Description for Affine Sets**: A is affine iff,

Affine sets: Dual Description

- **Dual Description for Affine Sets**: A is affine iff, $A = \{\mathbf{x} | P\mathbf{x} = b\}$ i.e. solution set of linear equations represented by $P\mathbf{x} = \mathbf{b}$ for some matrix P with rank $= \underline{n - dim(S)}$ and b
  - No Solution: $\mathbf{x} = \phi$. Is that affine? when b does not lie in the column space of P
  - Unique Solution: $\mathbf{x}$ is a point. when b lies in the column space of P and P is full rank
  - Infinitely Many Solutions: $\mathbf{x}$ is a line, or a plane, etc. when b lies in the column space of P and P is NOT full rank

(conversely every affine set can be expressed as solution set of system of linear equations)

When P has rank = 1, then A is a hyperplane

Convex Sets

Convex sets

- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

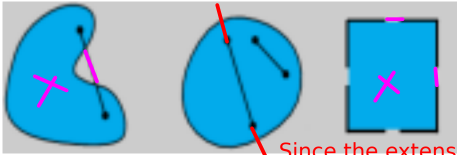
Convex set

- **In 2D, a line segment** between distinct points $\mathbf{x}_1$,$\mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.

$$\begin{aligned} \mathrm{x} = & \quad \alpha x_1 + \beta x_2 \\ \text{where} & \quad \alpha + \beta = 1, 0 \leq \alpha \leq 1 (\textit{also}, 0 \leq \beta \leq 1). \end{aligned}$$

Convex combination

- **Convex set** : $\mathbf{x}_1, \mathbf{x}_2 \in C, 0 \leq \alpha \leq 1 \Rightarrow \alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \in C$



The set in the centre is convex since convex combinations of (line segment joining) any two points lies in the set

Since the extensions (lines) do not belong to the set, it is not affine

  - Convex set is connected. Convex set can but not necessarily contains 'O'

Is every affine set convex? Is the reverse true?

YES

NO