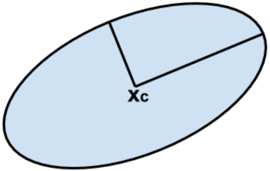
Euclidean balls and ellipsoids

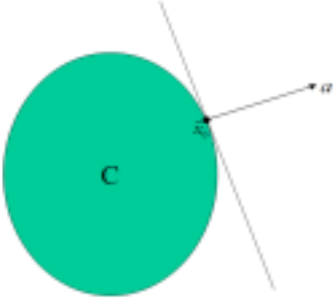
- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1\}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1}(\mathbf{x} - \mathbf{x}_c) \leq 1\}$, where P $\in S^n_{++}$ i.e. P is positive-definite matrix.
  - Other representation: $\{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\}$ with A square and non-singular (*i.e.*, A^{-1} exists).



Supporting hyperplane theorem and **Dual (H) Description**

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if

there exists an open ball around the point x
that lies completely within S

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points that are interior points

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points lying in the interior of $\mathcal{S}$ is denoted by $int(\mathcal{S})$ and is called the *interior* of $\mathcal{S}$. That is,

$$int(\mathcal{S}) = \left\{\mathbf{x} | \exists \epsilon > 0 \ s.t. \ \mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}\right\}$$

In the $1-$D case, the open interval obtained by excluding endpoints from an interval $\mathcal{I}$ is the interior of $\mathcal{I}$, denoted by $int(\mathcal{I})$. For example, $int([a, b]) = (a, b)$ and $int([0, \infty)) = (0, \infty)$.

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

Note: A set S may not contain its boundary
A ball around any boundary point should contain both points in the interior of the set as well as points outside

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

$$\partial(\mathcal{S}) = \left\{ \mathbf{y} | \forall\ \epsilon > 0,\ \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset \textit{ and } \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S}^C \neq \emptyset \right\}$$

For example, $\partial([a, b]) = \{a, b\}$. Boundary need not be contained in the set

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when, S coincides with its interior : int(S) = S

Recall Basic Prerequisite Concepts (in $\Re^n$)

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

$$\partial(\mathcal{S}) = \left\{\mathbf{y} | \forall\ \epsilon > 0,\ \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset \textit{ and } \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S}^C \neq \emptyset\right\}$$

For example, $\partial([a, b]) = \{a, b\}$.

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when, for every $\mathbf{x} \in \mathcal{S}$, there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}$.

1. The simplest examples of an open set are the open ball, the empty set $\emptyset$ and $\Re^n$.
2. Further, arbitrary union of opens sets is open. Also, finite intersection of open sets is open.
3. The interior of any set is always open. It can be proved that a set $\mathcal{S}$ is open if and only if $int(\mathcal{S}) = \mathcal{S}$.

Recall Basic Prerequisite Concepts (in $\Re^n$)

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when bnd(S) is contained in S

Recall Basic Prerequisite Concepts (in $\Re^n$)

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

The closed ball, the empty set $\emptyset$ and $\Re^n$ are three simple examples of closed sets. Arbitrary intersection of closed sets is closed. Furthermore, finite union of closed sets is closed.

Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by *closure*$(\mathcal{S})$ is given by

union of the set and bnd(S)

Recall Basic Prerequisite Concepts (in $\Re^n$)

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

The closed ball, the empty set $\emptyset$ and $\Re^n$ are three simple examples of closed sets. Arbitrary intersection of closed sets is closed. Furthermore, finite union of closed sets is closed.

Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by *closure*$(\mathcal{S})$ is given by

$$\mathit{closure}(\mathcal{S}) = \left\{\mathbf{y} \in \Re^n | \forall\, \epsilon > 0, \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset\right\}$$

Homework: Separating and Supporting Hyperplane Theorems (Fill in the Blanks)

SHT: Separating hyperplane theorem (a fundamental theorem)

If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, *i.e.*, $\mathcal{C} \cap \mathcal{D} = \phi$, then there exists $\mathbf{a} \neq \mathbf{0}$ and $b \in \Re$ such that
$\mathbf{a}^T\mathbf{x} \leq b$ for $\mathbf{x} \in \mathcal{C}$,
$\mathbf{a}^T\mathbf{x} \geq b$ for $\mathbf{x} \in \mathcal{D}$.
That is, the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{C}$ and $\mathcal{D}$.

- The seperating hyperplane need not be unique though.
- Strict separation requires additional assumptions (e.g., C is closed, D is a singleton).

Both separating and supporting hyperplane theorems become
trivial in the case of sets with empty interiors. Why?
1) Hyperplanes in R^n were defined as affine hulls of n affinely indepedent
points ==> Hyperplane has one dimension less than the space
2) If space is R^3, C and D are discs in R^2 lying on plane, trivial hyperplane (separating
or supporting) is the hyperplane containing C (and D) with equality everywhere

Proof of the Separating Hyperplane Theorem

Sum/difference of any two convex sets is convex

We first note that the set $\mathcal{S} = \{\mathbf{x} - \mathbf{y} | \mathbf{x} \in \mathcal{C}, \mathbf{y} \in \mathcal{D}\}$ is convex, since it is the sum of two convex sets. Since $\mathcal{C}$ and $\mathcal{D}$ are disjoint, $\mathbf{0} \notin \mathcal{S}$. Consider two cases:

1. Suppose $\mathbf{0} \notin closure(\mathcal{S})$. Let $\mathcal{E} = \{0\}$ and $\mathcal{F} = closure(\mathbf{S})$. Then, the euclidean distance between $\mathcal{E}$ and $\mathcal{F}$, defined as
   $dist(\mathcal{E};\mathcal{F}) = inf\{||\mathbf{u} - \mathbf{v}||_2 | \mathbf{u} \in \mathcal{E}, \mathbf{v} \in \mathcal{F}\}$
   is positive, and there exists a point $\mathbf{f} \in \mathcal{F}$ that achieves the minimum distance, i.e., $||\mathbf{f}||_2 = dist(\mathcal{E},\mathcal{F})$. Define a = f, b = 1/2 ||f||^2.
   Then $\mathbf{a} \neq \mathbf{0}$ and the affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b = \mathbf{f}^T(\mathbf{x} - \frac{1}{2}\mathbf{f})$ is nonpositive on $\mathcal{E}$ and nonnegative on $\mathcal{F}$, *i.e.*, that the hyperplane $\left\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = b\right\}$ separates $\mathcal{E}$ and $\mathcal{F}$. Thus, $\mathbf{a}^T(\mathbf{x} - \mathbf{y}) > 0$ for all $\mathbf{x} - \mathbf{y} \in \mathcal{S} \subseteq closure(\mathcal{S})$, which implies that, $\mathbf{a}^T\mathbf{x} \geq \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and $\mathbf{y} \in \mathcal{D}$.

Proof of the Separating Hyperplane Theorem

We first note that the set $\mathcal{S} = \left\{\mathbf{x}-\mathbf{y}|\mathbf{x}\in\mathcal{C},\mathbf{y}\in\mathcal{D}\right\}$ is convex, since it is the sum[6] of two convex sets. Since $\mathcal{C}$ and $\mathcal{D}$ are disjoint, $\mathbf{0}\notin\mathcal{S}$. Consider two cases:

1. Suppose $\mathbf{0}\notin \mathit{closure}(\mathcal{S})$. Let $\mathcal{E}=\{0\}$ and $\mathcal{F}=\mathit{closure}(\mathbf{S})$. Then, the euclidean distance between $\mathcal{E}$ and $\mathcal{F}$, defined as
   $\mathit{dist}(\mathcal{E};\mathcal{F}) = \mathit{inf}\left\{||\mathbf{u}-\mathbf{v}||_2|\mathbf{u}\in\mathcal{E},\mathbf{v}\in\mathcal{F}\right\}$
   is positive, and there exists a point $\mathbf{f}\in\mathcal{F}$ that achieves the minimum distance, i.e., $||\mathbf{f}||_2 = \mathit{dist}(\mathcal{E},\mathcal{F})$. Define $\mathbf{a}=\mathbf{f},\ b=1/2||\mathbf{f}||_2^2$.
   Then $\mathbf{a}\neq\mathbf{0}$ and the affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x}-b = \mathbf{f}^T(\mathbf{x}-\frac{1}{2}\mathbf{f})$ is nonpositive on $\mathcal{E}$ and nonnegative on $\mathcal{F}$, *i.e.*, that the hyperplane $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x}=b\right\}$ separates $\mathcal{E}$ and $\mathcal{F}$. Thus, $\mathbf{a}^T(\mathbf{x}-\mathbf{y})>0$ for all $\mathbf{x}-\mathbf{y}\in\mathcal{S}\subseteq \mathit{closure}(\mathcal{S})$, which implies that, $\mathbf{a}^T\mathbf{x}\geq\mathbf{a}^T\mathbf{y}$ for all $\mathbf{x}\in\mathcal{C}$ and $\mathbf{y}\in\mathcal{D}$.

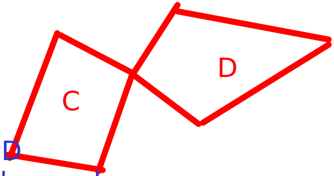
Nontrivial part, sketched on the board in class

[6] Easy proof. Let us attempt.

Proof of the Separating Hyperplane Theorem

That is, C and D are touching at a point...

2. Suppose, $0 \in closure(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $int(\mathcal{S}) = \emptyset$ (that is, if $\mathcal{S}$ has empty interior), it must lie in an affine set of dimension $< n$, and any hyperplane containing that affine set contains $\mathcal{S}$ and is a hyperplane.
   - In other words, $\mathcal{S}$ is contained in a hyperplane $\{\mathbf{z} | \mathbf{a}^T\mathbf{z} = b\}$, which must include the origin and therefore $b = 0$. In other words, $\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$ and all $\mathbf{y} \in \mathcal{D}$ gives us a trivial separating hyperplane.



Basically
the hyperplane
containing C and D
is the separating hyperplane
with a trivial equality everywhere on C and D

Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in closure(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has a nonempty interior, consider the set $\mathcal{S}_{-\epsilon} = \{\mathbf{z} | B(\mathbf{z}, \epsilon) \subseteq \mathcal{S}\}$ where $B(\mathbf{z}, \epsilon)$ is the Euclidean ball with center $\mathbf{z}$ and radius $\epsilon > 0$. $\mathcal{S}_{-\epsilon}$ is the set $\mathcal{S}$, shrunk by ϵ. $closure(\mathcal{S}_{-\epsilon})$ is closed and convex, and does not contain $\mathbf{0}$, so as argued before, it is separated from $\{\mathbf{0}\}$ by atleast one hyperplane with normal vector $\mathbf{a}(\epsilon)$ such that as in case 1
     a(eps)^T x >=0 is a separating hyperplane

     Without loss of generality assume $||\mathbf{a}(\epsilon)||_2 = 1$. Let ϵ_k, for $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim_{k \to \infty} \epsilon_k = 0$. Since $||\mathbf{a}(\epsilon_k)||_2 = 1$ for all k, the sequence $\mathbf{a}(\epsilon_k)$ contains a convergent subsequence, and let $\bar{\mathbf{a}}$ be its limit. We have

     ---

     which means $\bar{\mathbf{a}}^T \mathbf{x} \geq \bar{\mathbf{a}}^T \mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$, and $\mathbf{y} \in \mathcal{D}$.

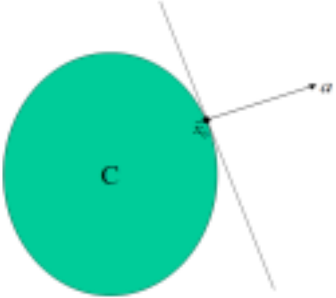
Proof of the Separating Hyperplane Theorem

2. Suppose, $0 \in \textit{closure}(\mathcal{S})$. Since $0 \notin \mathcal{S}$, it must be in the boundary of $\mathcal{S}$.
   - If $\mathcal{S}$ has a nonempty interior, consider the set
     $\mathcal{S}_{-\epsilon} = \left\{\mathbf{z} | B(\mathbf{z}, \epsilon) \subseteq \mathcal{S}\right\}$
     where $B(\mathbf{z}, \epsilon)$ is the Euclidean ball with center $\mathbf{z}$ and radius $\epsilon > 0$. $\mathcal{S}_{-\epsilon}$ is the set $\mathcal{S}$, shrunk by ϵ. $\textit{closure}\left(\mathcal{S}_{-\epsilon}\right)$ is closed and convex, and does not contain $\mathbf{0}$, so as argued before, it is separated from $\{\mathbf{0}\}$ by atleast one hyperplane with normal vector $\mathbf{a}(\epsilon)$ such that
     $\mathbf{a}(\epsilon)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{\epsilon}$
     Without loss of generality assume $||\mathbf{a}(\epsilon)||_2 = 1$. Let ϵ_k, for $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim_{k \to \infty} \epsilon_k = 0$. Since $||\mathbf{a}(\epsilon_k)||_2 = 1$ for all k, the sequence $\mathbf{a}(\epsilon_k)$ contains a convergent subsequence, and let $\bar{\mathbf{a}}$ be its limit. We have
     $\mathbf{a}(\epsilon_k)^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}_{-\epsilon_k}$ and therefore $\bar{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \textit{interior}(\mathcal{S})$, and $\bar{\mathbf{a}}^T\mathbf{z} \geq 0$ for all $\mathbf{z} \in \mathcal{S}$,
     which means $\bar{\mathbf{a}}^T\mathbf{x} \geq \bar{\mathbf{a}}^T\mathbf{y}$ for all $\mathbf{x} \in \mathcal{C}$, and $\mathbf{y} \in \mathcal{D}$.

Supporting hyperplane theorem (consequence of separating hyperplane theorem)

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point $\mathbf{x}$ of C.

HW: Proof of Supporting Hyperplane Theorem

The supporting hyperplane theorem is proved from the separating hyperplane theorem as follows:

1. If $int(C) \neq \emptyset$, the result follows by applying the separating hyperplane theorem to the sets $\{\mathbf{x}\}$ and $int(C)$.
2. If $int(C) = \emptyset$, then C must lie in an affine set of dimension $< n$, and any hyperplane containing that affine set contains C and $\mathbf{x}$, and is therefore a (trivial) supporting hyperplane.

Homework: Separating and Supporting Hyperplane Theorems (Fill in the Blanks Concluded)

Back to Euclidean balls and ellipsoids

- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1\}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\}$, where P $\in S^n_{++}$ i.e. P is positive-definite matrix.
  - Other representation: $\{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\}$ with A square and non-singular (*i.e.*, A^{-1} exists).

Back to Euclidean balls and ellipsoids

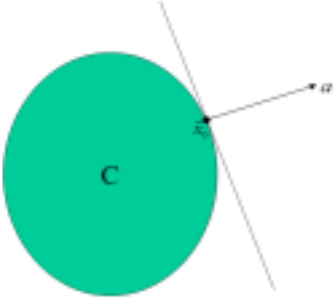
- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1\ \}$
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\ \}$, where P $\in S^n_{++}$ i.e. P is positive-definite matrix.
  - Other representation: $\{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\|_2 \leq 1\}$ with A square and non-singular (*i.e.*, A^{-1} exists).

Dual (H) Description for such convex sets?

Supporting hyperplane theorem and **Dual (H) Description**

Supporting hyperplane to set $\mathcal{C}$ at boundary point $\mathbf{x}_o$:

- $\left\{\mathbf{x}|\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_o\right\}$
- where $\mathbf{a} \neq 0$ and $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_o$ for all $\mathbf{x} \in \mathcal{C}$



Supporting hyperplane theorem: if $\mathcal{C}$ is convex, then there exists a supporting hyperplane at every boundary point of C.

Separating hyperplane and the Ellipsoid: Ellipsoid Algorithm

Implicit assumption on boundedness of set C is made

$$\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1 \}$$

- Given an ellipsoid $(P(i), \mathbf{x}_c(i))$ containing a set C Imagine C to be a polytope
- Ask a separating oracle to answer if $\mathbf{x}_c(i) \in C$ or compute separating hyperplane $\mathbf{a}, b$ between $\mathbf{x}_c(i)$ and C.
- If $\mathbf{x}_c(i) \notin C$, update ellipsoid center $\mathbf{x}_c(i+1)$ and ellipsoid shape $P(i)$

Examples of separating hyperplanes: (1) Cutting plane (cutting plane algos)
(2) Hyperplane based on gradient or subgradier
(we wil see 2 very soon)
(3) Say C = {x | Ax <= d} as in Linear Programs

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$. matrix induced vector norm
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$. L1, L infinity norm balls.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$ maximum norm of linear combination of columns of N subject to coefficients of x being bounded by the same norm

Here, $\sup\limits_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.

  - Eg: $M_N(I) =$

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} \| \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum\limits_{i=1}^{n} |a_{ij}|$ maximum column sum
  - If $N = \|.\|_2$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum\limits_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$,

Norm balls

- **Recap Norm:** A function[7] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$, $M_N(A) = \max_i \sum_{j=1}^{m} |a_{ij}|$