Closure under Affine transform (contd.)

In the finite-dimensional case each affine transformation is given by a matrix A and a vector $\mathbf{b}$. The image and pre-image of convex sets under an affine transformation defined as

$$f(\mathbf{x}) = \sum_{i}^{n} x_i a_i + b$$

yield convex sets[9]. Here a_i is the i^{th} row of A. The following are examples of convex sets that are either images or inverse images of convex sets under affine transformations:

1. the solution set of linear matrix inequality $(A_i, B \in \mathcal{S}^m)$

$$\left\{\mathbf{x} \in \Re^n \mid x_1 A_1 + \ldots + x_n A_n \preceq B\right\}$$

is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[10]. This set is the inverse image under an affine mapping of the Positive semi-definite cone?

[9] Exercise: Prove.

[10] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}_+^n$.

Closure under Affine transform (contd.)

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   is a convex set. Here $A \preceq B$ means $B - A$ is positive semi-definite[10]. This set is the inverse image under an affine mapping of the positive semi-definite cone. That is, $f^{-1}(cone) = \left\{\mathbf{x} \in \Re^n \,|B - (x_1 A_1 + \ldots + x_n A_n) \in \mathcal{S}_+^m\right\} = \left\{\mathbf{x} \in \Re^n |B \succeq (x_1 A_1 + \ldots + x_n A_n)\right\}$.

[9] Exercise: Prove.

[10] The inequality induced by positive semi-definiteness corresponds to a generalized inequality $\preceq_K$ with $K = \mathcal{S}_+^n$.

Closure under Affine transform (contd.)

2. hyperbolic cone which is the inverse image of the norm cone $\mathcal{C}_{m+1} = \{(\mathbf{z}, u) | ||\mathbf{z}|| \leq u, u \geq 0, \mathbf{z} \in \Re^m\} = \left\{(\mathbf{z}, u) | \mathbf{z}^T\mathbf{z} - u^2 \leq 0,\ u \geq 0, \mathbf{z} \in \Re^m\right\}$ is a convex set. The inverse image is given by

Closure under Affine transform (contd.)

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$$f^{-1}(\mathcal{C}_{m+1}) = \left\{\mathbf{x} \in \Re^n \mid \left(A\mathbf{x}, \mathbf{c}^T\mathbf{x}\right) \in \mathcal{C}_{m+1}\right\} = \left\{\mathbf{x} \in \Re^n | \mathbf{x}^T A^T A\mathbf{x} - (\mathbf{c}^T\mathbf{x})^2 \leq 0\right\}.$$
Setting $P = A^T A \in \mathcal{S}_+^n$, we get the equation of the hyperbolic cone

Closure under Affine transform (contd.)

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$$\left\{\mathbf{x} \mid \mathbf{x}^T P\mathbf{x} \leq (\mathbf{c}^T\mathbf{x})^2, \mathbf{c}^T\mathbf{x} \geq 0\right\}$$

Closure under Perspective and linear-fractional functions

The perspective function $P : \Re^{n+1} \to \Re^n$ is defined as follows:

$$P : \Re^{n+1} \to \Re^n \text{ such that}$$
$$P(x,t) = x/t \qquad \textbf{dom } P = \{(x,t) \mid t > 0\} \tag{40}$$

The linear-fractional function f is a generalization of the perspective function and is defined as: $\Re^n \to \Re^m$:

An affine transform in num/denom and then a perspective

$$f : \Re^n \to \Re^m \text{ such that}$$
$$f(\mathbf{x}) = \boxed{\frac{\mathbf{Ax}+\mathbf{b}}{\mathbf{c}^T\mathbf{x}+d}} \qquad \textbf{dom } f = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} + d > 0\} \tag{41}$$

The images and inverse images of convex sets under perspective and linear-fractional functions are convex[11].

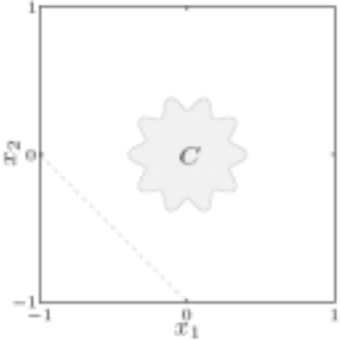
[11] Exercise: Prove.

Closure under Perspective (contd)

The Figure below shows an example perspective transform (3-D to 2-D effect)

Closure under linear-fractional functions (contd)

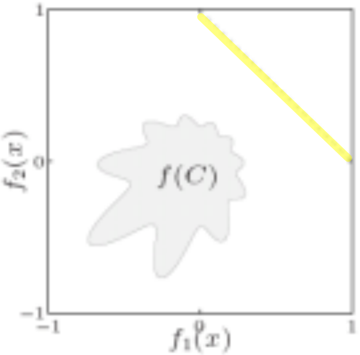
The Figure below shows an example set.



A set in R^2==> Apply affine transform to take it to R^3 ==> Apply perspective to get back to R^2

Closure under linear-fractional functions (contd)

Consider the linear-fractional function $f = \frac{1}{x_1+x_2+1}x$. The following Figure shows the image of the set (from the prevous slide) under the linear-fractional function f.



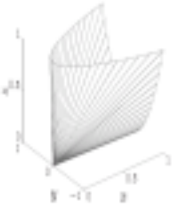
HW: $N = \|.\|_\infty$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})} = \sup_{\|\mathbf{x}\|=1} N(A\mathbf{x})$

1. If $N(\mathbf{x}) = \max_i |x_i|$ then $N(A\mathbf{x}) = \max_i |\sum_{j=1}^m a_{ij}x_j| \leq \max_i \sum_{j=1}^m |a_{ij}||x_j| \leq\leq \max_i \sum_{j=1}^m |a_{ij}|$ where the last inequality is attained by considering a $\mathbf{x} = [1, 1..1, 1...1]$ which has 1 in all positions. Then $\|\mathbf{x}\|_\infty = 1$ and for such an $\mathbf{x}$, the upper bounded on the supremum in indeed attained.
2. Therefore, it must be that $\|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^m |a_{ij}|$ (the maximum absolute row sum)
3. That is,

$$M_N(A) = \|A\mathbf{x}\|_1 = \max_i \sum_{j=1}^m |a_{ij}|$$

Max absolute row sum...

Another note Positive semidefinite cone & **Primal Description**



Consider symmetrix positive semi-definite matrix $S \in \Re^2$. Then S must be of the form

$$S = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \tag{42}$$

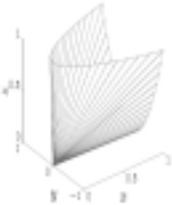
- We can represent the space of matrices $\mathcal{S}_+^2$ in $\Re^3$ with non-negative x and z coordinates and a non-negative determinant: Why?

Non-negative eigenvalues

t

sum of eigenvalues = trace   product of eigenvalues = determinan

Another note Positive semidefinite cone & **Primal Description**



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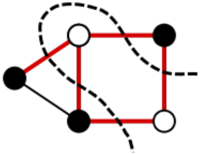
- We can represent the space of matrices $\mathcal{S}_+^2$ in $\Re^3$ with non-negative x and z coordinates and a non-negative determinant: Why?
- Sum of eigenvalues of a matrix is trace of matrix and product of its eigenvalues is its determinant.

Example Optimization Problem posed using Positive semidefinite Cone



- Consider the Max-Cut problem: Given a graph $G = (V, E)$ consisting of vertices V and edges E, partition V into subsets P and Q such that the cut-set (edges with one end-point in each set of the partition) has the maximum size.
- This can be posed as the following optimization problem:

$$\begin{aligned} &\text{maximize} && \sum_{(u,v)\in E} \frac{1 - x_u x_v}{2} && \text{Size of cut} \\ &\text{subject to} && x_u^2 = 1 && \text{for each } u \in V \end{aligned} \tag{43}$$

Ensures that each x_u is +1 or -1

Example Optimization Problem posed using Positive semidefinite Cone

- Again consider the optimization problem in (43) slightly rewritten in (44):

$$\begin{aligned} &\text{maximize} && \sum_{(u,v)\in E} \frac{1-\rho_{uv}}{2} && \\ &\text{subject to} && \rho_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle && \text{for all } u, v \in V \text{ and } u \neq v \\ & && \rho_{uu} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 && \end{aligned} \tag{44}$$

Here, $\mathbf{x}_*$ could be a scalar as in (43) or could be a vector in $\Re^n$ as well.

The problem in the second case
is no longer exactly equal to the
previous setting in (43)
It is an approximation
it turns out to be a best approximatior

Example Optimization Problem posed using Positive semidefinite Cone

- Again consider the optimization problem in (43) slightly rewritten in (44):

$$\begin{aligned} &\text{maximize} && \sum_{(u,v)\in E} \frac{1-\rho_{uv}}{2} \\ &\text{subject to} && \rho_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \qquad \text{for all } u, v \in V \text{ and } u \neq v \\ &&& \rho_{uu} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 \end{aligned} \tag{44}$$

Here, $\mathbf{x}_*$ could be a scalar as in (43) or could be a vector in $\Re^n$ as well.

- Consider the matrix $\rho \in \Re^{|V|\times|V|}$.

$$\rho = \begin{bmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \langle \mathbf{x}_1, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_1, \mathbf{x}_{|V|} \rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle & \langle \mathbf{x}_2, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_2, \mathbf{x}_{|V|} \rangle \\ \ldots. & \ldots. & \ldots. & \ldots. \\ \langle \mathbf{x}_{|V|}, \mathbf{x}_1 \rangle & \langle \mathbf{x}_{|V|}, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_{|V|}, \mathbf{x}_{|V|} \rangle \end{bmatrix}$$

This matrix is always symmetric and positive semi-definite

Example Optimization Problem posed using Positive semidefinite Cone

- Again consider the optimization problem in (43) slightly rewritten in (44):

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\text{maximize} \quad & \sum_{(u,v)\in E} \frac{1-\rho_{uv}}{2} \\
\text{subject to} \quad & \rho_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \qquad \text{for all } u,v \in V \text{ and } u \neq v \\
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  Here, $\mathbf{x}_*$ could be a scalar as in (43) or could be a vector in $\Re^n$ as well.
- Consider the matrix $\rho \in \Re^{|V|\times|V|}$.

$$\rho = \begin{bmatrix} \langle \mathbf{x}_1, \mathbf{x}_1 \rangle & \langle \mathbf{x}_1, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_1, \mathbf{x}_{|V|} \rangle \\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle & \langle \mathbf{x}_2, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_2, \mathbf{x}_{|V|} \rangle \\ \ldots. & \ldots. & \ldots. & \ldots. \\ \langle \mathbf{x}_{|V|}, \mathbf{x}_1 \rangle & \langle \mathbf{x}_{|V|}, \mathbf{x}_2 \rangle & \ldots & \langle \mathbf{x}_{|V|}, \mathbf{x}_{|V|} \rangle \end{bmatrix}$$

  This matrix is symmetric and positive semi-definite.
- The ellipsoid algorithm (outlined in a previous lecture) can solve SDP in polytime and

Convex Functions, Epigraphs, Sublevel sets, Separating and Supporting Hyperplane Theorems and required tools

Convex Functions: Extending Slopeless Definition from $\Re :\to \Re$

Convex Functions: Extending Slopeless Definition from $\Re :\to \Re$

- A function $f : \mathcal{D} \to \Re$ is **convex** if

D is convex
and
f is such that the line segment connecting two
points on the function curve always lies
above the function curve itself

Convex Functions: Extending Slopeless Definition from $\Re :\to \Re$

- A function $f : \mathcal{D} \to \Re$ is **convex** if $\mathcal{D}$ is a convex set and

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{45}$$

- A function $f : \mathcal{D} \to \Re$ is **strictly convex** if $\mathcal{D}$ is convex and

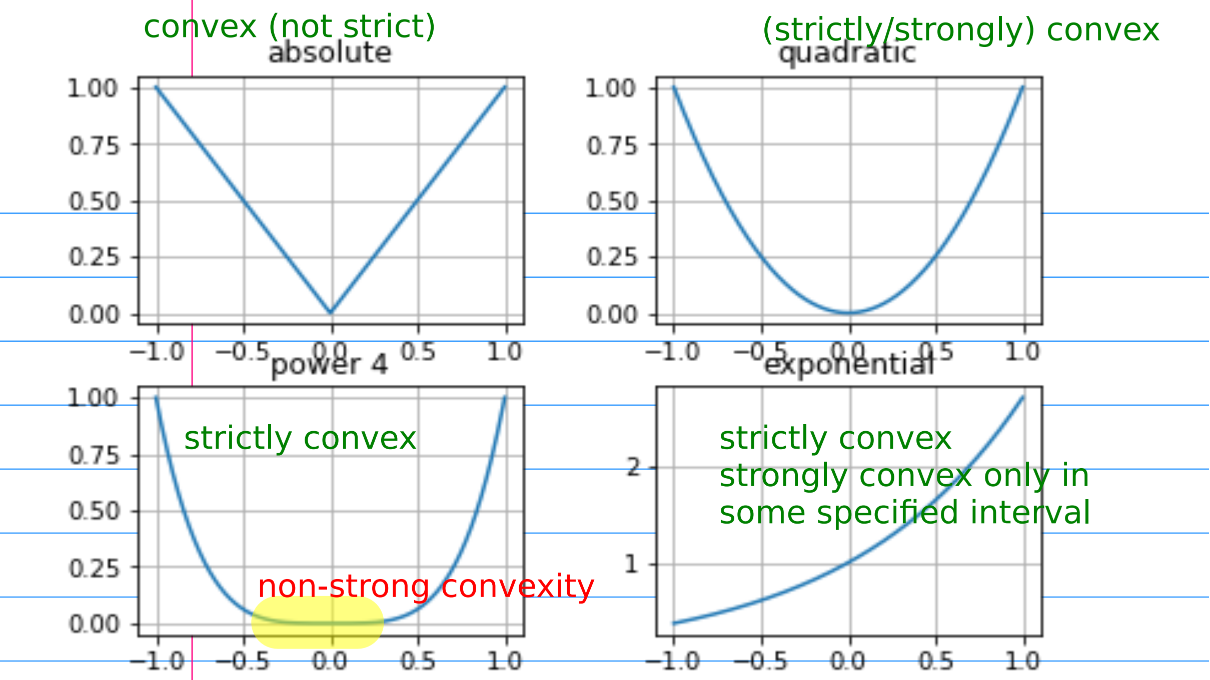
$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) < \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 < \theta < 1 \tag{46}$$

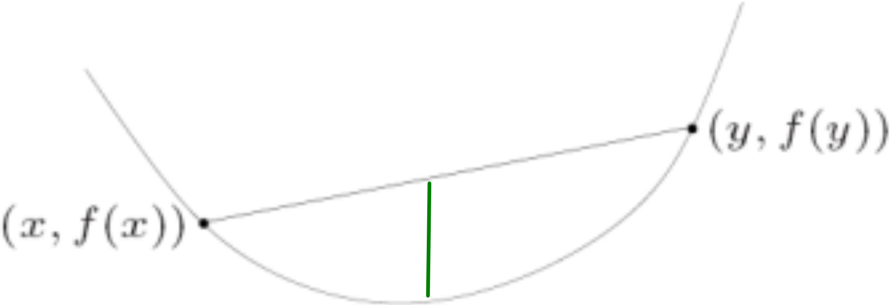
- A function $f : \mathcal{D} \to \Re$ is **strongly convex** if $\mathcal{D}$ is convex and for some constant $c > 0$

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) - \frac{1}{2} c\theta(1-\theta)||\mathbf{x} - \mathbf{y}||^2 \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1$$

- A function $f : \mathcal{D} \to \Re$ is **uniformly convex** wrt function $d(\mathbf{x}) \geq 0$ (vanishing only at 0) if $\mathcal{D}$ is convex and

$$f(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y})) - d(||\mathbf{x} - \mathbf{y}||)\theta(1-\theta) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1$$


convex (not strict)
absolute
(strictly/strongly) convex
quadratic
power 4
strictly convex
non-strong convexity
exponential
strictly convex
strongly convex only in
some specified interval
1.00
0.75
0.50
0.25
0.00
−1.0
−0.5
0.0
0.5
1.0
2
1



Strong convexity is about characterizing gap as a function of ||x-y||

Figure 13: Example of convex function.

Examples of Convex Functions

Examples of convex functions on the set of reals $\Re$ as well as on $\Re^n$ and $\Re^{m\times n}$ are shown below.

Function type	Domain	Additional Constraints
The affine function: $ax+b$	$\Re$	Any $a, b \in \Re$
The exponential function: e^{ax}	$\Re$	Any $a \in \Re$
Powers: x^α	$\Re_{++}$	$\alpha \geq 1$ or $\alpha \leq 1$
Powers of absolute value: $\lvert x\rvert^p$	$\Re$	$p \geq 1$
Negative entropy: $x \log x$	$\Re_{++}$	
Affine functions of vectors: $\mathbf{a}^T\mathbf{x}+b$	$\Re^n$	
p-norms of vectors: $\lvert\lvert\mathbf{x}\rvert\rvert_p = \left(\sum_{i=1}^{n} \lvert x_i\rvert^p\right)^{1/p}$	$\Re^n$	$p \geq 1$
inf norms of vectors: $\lvert\lvert\mathbf{x}\rvert\rvert_\infty = \max_k \lvert x_k\rvert$	$\Re^n$	
Affine functions of matrices: $tr(A^TX)+b = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}X_{ij}+b$	$\Re^{m\times n}$	
Spectral (maximum singular value) matrix norm: $\lvert\lvert X\rvert\rvert_2 = \sigma_{max}(X) = (\lambda_{max}(X^TX))^{1/2}$	$\Re^{m\times n}$	

Table 1: Examples of convex functions on $\Re$, $\Re^n$ and $\Re^{m\times n}$.

You could prove these convexities from first principles..
OR develop tools for convex functions (like for convex sets) and use them to prove convexity?

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**

General quadratic functions

Strict, Strong and Uniform Convexity for $f : \Re \rightarrow \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \rightarrow \Re$,

x^TAx + b^Tx + c

Strict, Strong and Uniform Convexity for $f: \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f: \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**

x^4, x^6...x raised to even powers?

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^8$
- **Convex but not Strictly Convex:**

  $|x|$

Strict, Strong and Uniform Convexity for $f: \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f: \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^8$
- **Convex but not Strictly Convex:**
  - $f(x) = |x|$

A function $f : \Re^n \rightarrow \Re$ is said to be concave if the function $-f$ is convex. Examples of concave functions on the set of reals $\Re$ are shown below. If a function is both convex and concave, it must be affine, as can be seen in the two tables.

H/w : prove in general

Function type	**Domain**	**Additional Constraints**
The affine function: $ax + b$	$\Re$	Any $a, b \in \Re$
Powers: x^α	$\Re_{++}$	$0 \leq \alpha \leq 1$
logarithm: $\log x$	$\Re_{++}$	

Table 2: Examples of concave functions on $\Re$.

Convexity and Global Minimum

Fundamental chracteristics:

1. Any point of local minimum point is also a point of global minimum.
2. For any stricly convex function, the point corresponding to the gobal minimum is also unique. we have already seen that a strongly convex function is strictly convex

To discuss these further, we need to extend the defitions of Local Minima/Maxima to arbitrary sets $\mathcal{D}$

Illustrating Local Extrema for $f: \Re^2 \rightarrow \Re$

These definitions are exactly analogous to the definitions for a function of single variable. Figure below shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

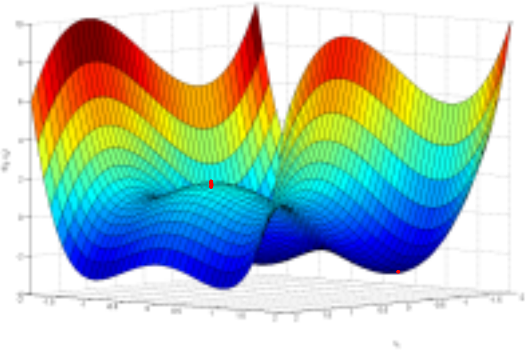


Figure 14:

Local Extrema in Normed Spaces: Extending from $\Re \to \Re$

Local Extrema in Normed Spaces: Extending from $\Re \rightarrow \Re$

Definition

[Local maximum]: *A function f of n variables has a local maximum at $\mathbf{x}^0 \in \mathcal{D}$ in a normed space $\mathcal{D}$ if $\exists \epsilon > 0$ such that $\forall$ $||\mathbf{x} - \mathbf{x}^0|| < \epsilon$. $f(\mathbf{x}) \leq f(\mathbf{x}^0)$. In other words, $f(\mathbf{x}) \leq f(\mathbf{x}^0)$ whenever $\mathbf{x}$ lies in the interior of some norm ball around $\mathbf{x}^0$.*

Definition

[Local minimum]: *A function f of n variables has a local minimum at $\mathbf{x}^0 \in \mathcal{D}$ in a normed space $\mathcal{D}$ if $\exists \epsilon > 0$ such that $\forall$ $||\mathbf{x} - \mathbf{x}^0|| < \epsilon$. $f(\mathbf{x}) \geq f(\mathbf{x}^0)$. In other words, $f(\mathbf{x}) \geq f(\mathbf{x}^0)$ whenever $\mathbf{x}$ lies in the interior of some norm ball around $\mathbf{x}^0$.*

1. These definitions can be easily extended to metric spaces or topological spaces. But we need recall definitions of open sets and interior in those spaces (and in fact some other foundations will also help).
2. We will first provide these defintions in $\Re^n$ and then provide the idea for extending them to more abstract topological/metric/normed spaces.

Positive Semidefinite Cone, Generalized Inequality and Convex Analysis

More on Convex Sets and Advanced Material on Convex Analysis

- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Linear program and dual of LP.
- Properties of dual cones.
- Conic Program.
- Generalized Inequalities.

Positive semidefinite cone: Notes

1. Claim : $(S^n_+)^* = (S^n_+)$
2. i.e. $<$X,Y$>$ = tr(X^TY) = tr(XY) $\geq 0 \ \forall$ X $\in (S^n_+)$ iff Y $\in (S^n_+)$

Proof:

1.
   1. Let us say Y $\notin S^n_+$. That is $\exists$ z $\in \Re^n$ s.t. $z^T Yz =$ tr$(zz^T Y) < 0$
   2. i.e. $\exists$ X $= zz^T \in S^n_+$ s.t. $<$X,Y$> < 0$
   3. $\implies$ Y $\notin (S^n_+)^*$
2.
   1. Suppose Y,X $\in S^n_+$. Any X $\in S^n_+$ can be written in terms of eignvalue decomposition as:
   2. X $= \sum_{i=1:n} \lambda_i u_i u_i^T$ $(\lambda_i \geq 0)$
   3. $\therefore$ $<$Y,X$>$ = tr(YX) = tr(Y $\sum_{i=1:n} \lambda_i u_i u_i^T) = \sum_{i=1:n} \lambda_i tr(Y u_i u_i^T) = \sum_{i=1:n} \lambda_i u_i^T Y u_i \geq 0$.
   4. Since $(\lambda_i \geq 0)$ and $(u_i^T Y u_i \geq 0$ as Y $\in S^n_+)$
   5. $\implies$ Y $\in (S^n_+)^*$

Positive semidefinite cone: Questions

1. Q) Is there some connection between Y $= yy^T$ used for $S^n_+ = \{X \in S^n \mid <yy^T, X> \geq 0\}$ and $(S^n_+)^* = (S^n_+)$.
   - (To be revisited as H/W)
2. Q) $(S^n_{++})^* = ?$, int$(S^n_+) = (S^n_{++})$
   - Ans: $(S^n_{++})^* = (S^n_+)$, (will be done formally for general case of convex cones)
   - C = convex cone, C^{**} = cl(C)
3. Q) Consider an application of psd cone for optimization. (thru LP)
   1. We will first see (weak) duality in a linear optimization problem (LP).
   2. Next we look at generalized (conic) inequalities and the properties that the cone must satisfy for the inequality to be a valid inequality.
   3. Next, we generalize LP to conic program (CP) using generalized inequality and realize weak duality for CP thru dual cones.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+b \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x}+b) \leq 0$
   - $\implies c^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x}+b)$
   - $\implies c^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $c^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,
   $$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T\mathbf{b}, & \text{if } A^T\lambda = c \\ -\infty, & \text{otherwise} \end{cases}$$
   - Note: LHS ($\mathbf{c}^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T\mathbf{b}$) is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded) $\geq$ Dual LP (upper bounded):
   $(\min_{\mathbf{x}\in\Re^n} \mathbf{c}^T\mathbf{x}$, s.t. $A\mathbf{x} \geq \mathbf{b}) \geq (max_{\lambda\geq 0}\mathbf{b}^T\lambda$, s.t. $A^T\lambda = \mathbf{c})$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$
   - Note: $-A\mathbf{x} + b \leq 0$ can be rewritten as $A\mathbf{x} \geq 0$.
   - So, constraint is $A\mathbf{x} - b \in R^n_+$
   - Note: R^n_+ is a CONE. How about defining generalized inequality for a cone K as: $c \geq_K d$ iff $c - d \in K$
2. So, a generalized conic program can be defined as:
   $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
   - That is, constraint is $A\mathbf{x} - b \in K$.

Properties of dual cones

1. If X is a Hilbert space & $C \subseteq X$ then C^* is a closed convex cone.
   - We have already proven that C^* is a closed convex cone.
   - C^* = intersection of infinite topological half spaces.
   - $C^* = \cap_{\mathbf{x} \in C} \{y | y \in X, < \mathbf{y}, \mathbf{x} > \geq 0\}$
   - $\implies C^*$ is closed.
2. $C_1 \subseteq C_2 \implies C_2^* \subseteq C_1^*$.
3. interior$(C^*) = \{\mathbf{y} \in X| < \mathbf{y}, \mathbf{x} >> 0\}$
4. If C is cone and has $int(C) \neq \emptyset$ then C^* is pointed.
   - Since; if $\mathbf{y} \in C^*$ & $-\mathbf{y} \in C^*$, then $\mathbf{y} = 0$.
5. If C is cone then closure$(C) = C^{**}$
   - If C = open half space, then C^{**} = closed half space.
6. If closure of C is pointed, then interior$(C^*) \neq \phi$.

S is called conically spanning set of cone K iff $conic(S) = K$.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)
  - i.e. K has no straight lines passing through O.
  - i.e. if $-a, a \in K$, then a $= 0$

examples

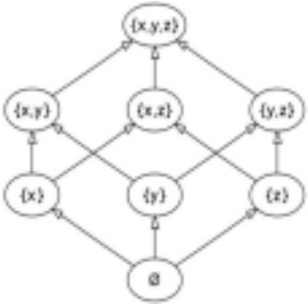
- non-negative orthant $K = R^n_+ = \{\mathbf{x} \in \Re^n | \mathbf{x}_i \geq 0, i = 1, ..., n\}$
- positive semidefinite cone $K = S^n_+$
- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

1. Reflexivity: $a \geq a$;
2. Anti-symmetry: if both $a \geq b$ and $b \geq a$, then $a = b$;
3. Transitivity: if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. Compatibility with linear operations:
   1. Homogeneity: If $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$, i.e. one can multiply both sides of an inequaility by a nonnegative real.
   2. Addititvity: if both $a \geq b$ abd $c \geq d$, then $a + c \geq b + d$, i.e. One can add two inequalities of the same sign.

Example of Partial Order

- Example of Partial Order $\subseteq$ over sets
- The Hasse diagram of the set of all subsets of a three-element set $\{\mathbf{x}, y, z\}$, ordered by inclusion(Inclusion, i.e. the Partial Order $\subseteq$):
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Dual Cones and Generalized Inequalities

Instructor: Prof. Ganesh Ramakrishnan

Contents: Vector Spaces beyond $\Re^n$

- Recap: Linear program (LP) & dual of LP.
- Recap: Conic program.
- Recap: Linear program (LP) & dual of LP.

Linear program (LP) & dual of LP.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x} \in \Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + \mathbf{b} \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in \Re^n_+$)
   - Then $\lambda^T(-A\mathbf{x} + \mathbf{b}) \leq 0$
   - $\implies \mathbf{c}^T\mathbf{x} \geq c^T\mathbf{x} + \lambda^T(-A\mathbf{x} + b)$
   - $\implies \mathbf{c}^T\mathbf{x} \geq \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - So, $\mathbf{c}^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T\mathbf{b} + (\mathbf{c} - A^T\lambda)^T\mathbf{x}$
   - Thus,
   $$\mathbf{c}^T\mathbf{x} \geq \begin{cases} \lambda^T b, & \text{if } A^T\lambda = \mathbf{c} \\ -\infty, & \text{otherwise} \end{cases}$$
   - Note: LHS ($c^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T b$) is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\geq$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x} \in \Re^n} \mathbf{c}^T\mathbf{x}, s.t. A\mathbf{x} \geq b) \geq (max_{\lambda \geq 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

We will motivate through linear programming (LP), generalized inequalities:

1. A generalized conic program can be defined as:
   $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$
   - That is, constraint is $A\mathbf{x} - b \in K$.
2. Q: Has to generalize $-A\mathbf{x} + b \leq 0$ to $-A\mathbf{x} + b \leq_K 0$ s.t. $\leq_K$ is a generalized inequality & K some set?
3. What properties should K satisfy so that $\leq_K$ satisfies properties of generalized inequalities?

Valid Inequality and Partial Order

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, reall that any partial order $\geq$ should satisfy the following properties:(refer page 51 of `www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf`):

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    1. Homogeneity: If $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$, i.e. one can multiply both sides of an inequaility by a nonnegative real.
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Example of Partial Order

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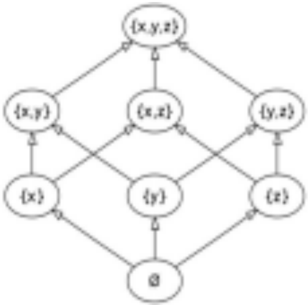
- 

- (source `http://en.wikipedia.org/wiki/Partially_ordered_set`)

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.

Proof:

1. K being pointed convex cone $\implies$ $\geq_K$ is a partial order
    1. Reflexivity: $a \geq_K a$, since $a - a = 0 \in K$ ($\because$ K is cone)
    2. Anti-symmetry: If a $\geq_K$ b & b $\geq_K$ a then a $=$ b, since a - b $\in$ K & b -a $\in$ K $\implies$ a - b $=$ 0 ($\because$ K is pointed)
    3. Transitivity: If both a $\geq_K$ b & b $\geq_K$ c then a $\geq_K$ c, since a - b $\in$ K & b -c $\in$ K $\implies$ (a - b) $+$ (b - c) $\in$ K ($\because$ K is a convex cone i.e. contain all conic combinations of points in the set)
    4. Homogeneity: If both a $\geq_K$ b & $\lambda \geq 0$ then λa $\geq_K$ λb, since a - b $\in$ K & $\lambda \geq 0$ $\implies$ λ(a - b) $\in$ K ($\because$ K is a cone)
    5. Additivity: If a $\geq_K$ b & c $\geq_K$ d then a $+$ c $\geq_K$ b $+$ d, since a - b $\in$ K & c -d $\in$ K $\implies$ (a $+$ c) - (b $+$ d) $\in$ K ($\because$ K is a convex cone)
2. $\geq_K$ is a partial order $\implies$ K being pointed convex cone

Proof of generalized inequality

To prove that K being closed, solid and pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality.
Proof:

1. $\geq_K$ is a partial order $\implies$ K being pointed convex cone
   1. K is convex cone: If $\mathbf{x}, \mathbf{y} \in K$ then $\theta_1\mathbf{x} + \theta_2 y \in K \forall\ \theta_1, \theta_2 \geq 0$, since $\mathbf{x} \geq_K 0$ & $y \geq_K 0 \implies \theta_1\mathbf{x} \geq_K 0$ & $\theta_2 y \geq_K 0\ \forall\ \theta_1, \theta_2 \geq 0$ (Homogeneity of $\geq_K$) and thus $\theta_1\mathbf{x} + \theta_2 y \geq 0$ (Additivity of $\geq_K$)
   2. K is pointed: If $\mathbf{x} \in K$ & $-\mathbf{x} \in K$ then $\mathbf{x} = 0$, since $\mathbf{x} \geq_K \mathbf{x}$ & $-\mathbf{x} \geq_K 0 \implies 0 \geq_K \mathbf{x}$ (reflectivity $\mathbf{x} \geq_K \mathbf{x}$, and adding $\mathbf{x} \geq_K \mathbf{x}$&$-\mathbf{x} \geq_K 0$ by additivity) and $-\mathbf{x} \geq_K \mathbf{x}$ (additivity on $-\mathbf{x} \geq_K 0$ & $0 \geq_K \mathbf{x}$) and similarly $\mathbf{x} \geq_K -\mathbf{x}$, and by applying anti-symmetry on $-\mathbf{x} \geq_K \mathbf{x}$ & $\mathbf{x} \geq_K -\mathbf{x}$ we get $\mathbf{x} = -\mathbf{x}$ i.e. $\mathbf{x} = 0$.

Additional properties over & above K being pointed convex cone

1. Que: Suppose $a^i \geq_K b^i$ $\forall$ i & $a^i \rightarrow a$ & $b^i \rightarrow b$, then for $a \geq_K b$ what more is required of K?
2. Ans: Necessary condition is that a^i - $b^i \rightarrow a - b \in$ K. i.e. K is closed(Also happens to be a sufficient condition).
3. Que: What is required so that $\exists$ $a >_K b$ (i.e. $b \not\geq_K a$)?
4. Ans: Sufficient condition is that $a - b \in$ int(K) i.e. int(K) $\neq \phi$ OR K has non-empty interior.

Linear program (LP) & Conic program.

We will first see (weak) duality in a linear optimization problem (LP).

1. LP: $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x} + b \leq 0$

$-A\mathbf{x} + b \leq 0$ can be rewritten as $A\mathbf{x} \geq b$ or $A\mathbf{x} - b \in \Re^n_+$ Note: $\Re^n_+$ is a CONE. How about defining generalized inequality for a cone C as c $>_K$ d iff $c - d \in K$ and a generl conic program as:

1. $\min_{\mathbf{x} \in \Re^n} c^T \mathbf{x}$
   subjected to $-A\mathbf{x} + b \leq_K 0$

- That is, constraint is $A\mathbf{x} - b \in K$.
- K is a proper cone.

Generalized Inequalities

a convex cone $K \subseteq \Re^n$ is a proper cone (or regular cone) if:
(Some restrictions on K that we will require, H/W Why?)

- K is closed (contains its boundary)
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- nonnegative polynomials on [0,1]:
  $K = \{\mathbf{x} \in \Re^n | x_1 + x_2 t + x_3 t^2 + ... + x_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\}$
- Que: What if $n \to \infty$, can you get proper cones under additional constraints?

Linear program & its dual To Conic program and its dual.

Consider LP and its dual:

1. LP: $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$ (Affine Objective)
   subjected to $-A\mathbf{x}+b \leq 0$
   - Let $\lambda \geq 0$ (i.e. $\lambda \in R^n_+$)
   - Then $\lambda^T(-A\mathbf{x}+b) \leq 0$
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   - So, $c^T\mathbf{x} \geq \min_{\mathbf{x}} \lambda^T b + (c - A^T\lambda)^T\mathbf{x}$
   - Thus,
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   - Note: LHS ($\mathbf{c}^T\mathbf{x}$) is independent of λ and R.H.S ($\lambda^T\mathbf{b}$) is independent of $\mathbf{x}$.
2. Weak duality theorem for Linear Program:
   Primal LP (lower bounded by dual) $\geq$ Dual LP (upper bounded by primal):
   $(\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}, s.t. A\mathbf{x} \geq b) \geq (max_{\lambda\geq 0} b^T\lambda, s.t. A^T\lambda = c)$

Conic program

Refer page 5 of `http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf`:

1. Conic program:
   $\min_{\mathbf{x}\in\Re^n} c^T\mathbf{x}$
   subjected to $-A\mathbf{x}+b \leq_K 0$
2. Generalized conic program:
   $\min_{\mathbf{x}\in V} < c, \mathbf{x} >_V$
   subjected to $A\mathbf{x} - b \in K$
3. K is a regular/proper cone.
4. We need an equivalent $\lambda \in D \supseteq K^*$ s.t.
   $<\lambda, A\mathbf{x} - b >\geq 0$.
5. This K^* s.t.
   $D = \{\lambda | < \lambda, A\mathbf{x} - \mathbf{b} >\geq 0, \; \lambda \in V \; \forall \; A\mathbf{x} - b \in K\}$
   & $D \supseteq K^*$ is dual cone of K

Dual of Conic program

1. Refer page 7 of `http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf`: $K^* = \{\lambda : \lambda^T \xi \geq 0 \ \forall \xi \in K\}$ is the cone dual to K.
2. With this follows weak duality theorem for CONIC PROGRAM:
   Primal CP (lower bounded by dual) $\geq$ Dual CP (upper bounded by primal):
   $(\min_{\mathbf{x} \in V} < c, \mathbf{x} >_V$, s.t. $< \lambda, A\mathbf{x} - \mathbf{b} > \geq 0.) \geq (max_{\lambda \in K^*} < b, \lambda >, s.t. A^T \lambda = c)$

Notes: LP and CP

1. Both LP and CP dealt with affine objectives.
2. CP dealt with the generalized conic inequalities.
3. Later, in convex optimization, we will deal with the more general convex functions in the objective.

Some Generalizations:

1. If $K = R^n_+$, the CP is an LP.
2. If $K = S^n_+$ (Set of all *nXn* SPD matrices), the CP is an SDP (Semi-definite program).
3. Any generic convex program can be expressed as a cone program (CP).

Dual of dual

1. If K is a closed convex cone then $K^{**} = K$.
2. More generally, if K is just a convex cone, $K^{**} =$ closure(K) (abbreviated as Cl(K))

We will prove that if K is closed, then $K^{**} = K$:

1. $K \subseteq K^{**}$, since $\mathbf{x} \in K \implies <\mathbf{x}, \mathbf{y}> \geq 0 \ \forall \ \mathbf{y} \in K^* \implies \mathbf{x} \in K^{**}$.
2. $K^{**} \subseteq K$, we will prove by contradiction. Suppose $\mathbf{x} \in K^{**}$ but $\mathbf{x} \notin K$:
    1. K^{**} is closed since any dual cone is intersection of half spaces that are closed.
    2. $\{\mathbf{x}\}$ is a singleton set.
    3. $\implies$ by "strict hyperplane theorem" (on next page and proved later): $\exists \mathbf{a} \in V \ \& \ \mathbf{b} \in \Re$ s.t. $<\mathbf{a}, \mathbf{x}> < \mathbf{b} \& <\mathbf{a}, \mathbf{y}> \geq b \forall \ \mathbf{y} \in K$.
    4. $\implies <\mathbf{a}, \mathbf{x}> < 0 \leq <\mathbf{a}, \mathbf{y}> \forall \mathbf{y} \in K$. (Since $\mathbf{y} = 0 \in K^{**}$, Claim: $\mathbf{b} = 0$ if V is a closed convex cone)
    5. $\implies \ \mathbf{a} \in K^* \ \& \ \mathbf{x} \notin K^{**}$ [contradiction]