Well-Ordered Sets and Generalized Induction

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1 Definitions

1.1 Partial Order

Given a set S, a relation $R \subseteq S \times S$ is called a *partial order* if

- R is reflexive, i.e., $\forall s \in S, (s,s) \in R$.
- R is anti-symmetric, i.e., $\forall s,t \in S$, if $(s,t) \in R$ and $s \neq t$, then $(t,s) \notin R$.
- R is transitive, i.e., $\forall s,t,u \in S$, if $(s,t) \in R$ and $(t,u) \in R$, then $(s,u) \in R$.

1.2 Total Order

Relation R is called a *total order* if

- R is a partial order.
- $\forall s,t \in S$, either (s,t) or (t,s) is in R.

1.3 Well Order

Relation R is called a *well order* if

1. R is a total order.
2. For every non-empty subset A of S, there exists an element $a \in A$ such that $\forall x \in A, (a,x) \in R$. The element a is called a *least element* of A.

Example: Let Z^+ be the set of non-negative integers, i.e., $Z^+ = \{0,1,2,\ldots\}$. This set is well-ordered by the $\leq$ relation. Consider the set $Z^+ \times Z^+ = \{(x,y) \mid x,y \in Z^+\}$. Define the extended $\leq$ relation on $Z^+ \times Z^+$ as $(a,b) \leq (x,y)$ if and only if one of the following conditions hold: (i) $a < x$ or (ii) $a = x$ and $b \leq y$. This defines a well-ordering of $Z^+ \times Z^+$.

Can you think of a well-ordering relation for the set of real numbers? To see why the $\leq$ relation doesn't work, consider the non-empty subset of all positive real nos (0 not included). There doesn't exist a least element (for $\leq$) for this subset. So the second condition of well ordering is violated.

2 Generalized Induction

Let S be a set with a well-ordering relation R. Let $P(x)$ be a statement about $x \in S$. Suppose we know that "if $P(x)$ holds for all x such that $(x, y) \in R$ and $x \neq y$, then $P(y)$ holds as well". By generalized induction, we can then say "$P(x)$ holds for all $x \in S$".

Proof: [On the lines of D. E. Knuth in "The Art of Computer Programming Vol 1"]

Let $A = \{x \mid x \in S \text{ and } P(x) \text{ does not hold}\}$. Our goal is to show that $A = \phi$. We prove this by contradiction. If possible, let A be non-empty. Clearly, $A \subseteq S$. Since S is well-ordered by R, there exists a least element $a \in A$ such that $(a, x) \in R$ for all $x \in A$. Note that since $a \in A$, $P(a)$ **does not hold** by definition.

Now let $B = \{x \mid x \in S, x \neq a \text{ and } (x, a) \in R\}$. Anti-symmetricity of R implies $A \cap B = \phi$. There are two cases to consider.

$B = \phi$: In this case, "$P(x)$ holds for all x such that $(x, a) \in R$ and $x \neq a$" is vacuously true. Consequently, "$P(a)$ holds as well". (Recall that if p implies q and p is true, then q must be true).

$B \neq \phi$: Since $A \cap B = \phi$, $P(x)$ holds for all $x \in B$. Therefore, by definition of B, "$P(x)$ holds for all x such that $(x, a) \in R$ and $x \neq a$". Consequently, "$P(a)$ holds as well".

In both cases, we conclude that $P(a)$ holds which contradicts our earlier assertion that $P(a)$ does not hold. Therefore, our premise that A is non-empty must be incorrect.