REFERENCES

[1] R. S. Kennedy *Fading Dispersive Communication Channels.* New York: Wiley, 1969.
[2] R. F. Daly "Signal Design for Efficient Detection in Dispersive Channels" *IEEE Trans. Inform. Theory,* 206–213, Mar. 1970.
[3] E. Mosca "Signal Selection in Transmitted Reference Communication Systems" 1972 Int. Conf. on Communications, Philadelphia, Penn., June 1972.
[4] E. Conte, M. Longo, and E. Mosca "A Procedure for Signal Selection for Digital Communications via Fading Dispersive Channels" submitted for publication to *Alta Frequenza.*
[5] P. A. Bello "Characterization of Randomly Time-Variant Linear Channels" *IEEE Trans. Communications Systems,* 360–393, Dec. 1963.
[6] J. N. Pierce "Theoretical Limitations on Frequency and Time Diversity for Fading Binary Transmissions," *IRE Trans. Comm. Systems,* 186–189, June 1961.
[7] H. L. Van Trees *Detection, Estimation and Modulation Theory,* vol. III, page 292, New York: Wiley, 1971.
[8] R. S. Kennedy and I. L. Lebow "Signal Design for Dispersive channels" *IEEE Spectrum,* 231–237, March 1967.
[9] P. Monsen "Digital Transmission Performance on Fading Dispersive Diversity Channels" *IEEE Trans. Communication Systems,* 33–39, Jan. 1973.
[10] S. K. Chow and A. N. Venetsanopoulos: "Optimal On-Off Signaling over Linear Time-Varying Stochastic Channels" *IEEE Trans. on Information Theory,* 602–609, September 1974.

Correspondence

Proof of a Conjecture on the Interarrival-Time Distribution in an $M/M/1$ Queue with Feedback

P. J. BURKE

***Abstract*—In a Jackson-type queuing network with feedback, the equilibrium state distribution of each queue is that of an $M/M/s$ system. In support of a previous conjecture that nevertheless the input processes in such a network are not Poisson, the marginal interarrival-time distribution for an equilibrium $M/M/1$ queuing system with feedback, counting both fed-back and exogenous customers as arrivals, is calculated. Since this distribution is a mixture of two exponentials, the total input to such a system is not Poisson.**

We consider a queuing system in which a customer (call) rejoins the queue with fixed probability p at the conclusion of his service. All distributions involved are equilibrium distributions. The calls originally arrive at the queue in a Poisson stream and are served by a single exponential server. The "total" input process (input) consists of the superposition of the stream of originally arriving (exogenous) calls and the stream of calls rejoining the queue. Since these are not independent, the statistical character of the input is not evident.

The state of the system is defined as the number of calls in the system waiting or being served. It is known that the state distribution of the queuing system described above is the same as that of an ordinary $M/M/1$ system with an enhanced traffic parameter. In fact, in a network of exponential-service multiserver queues with feedback, where the fed-back calls go to any queue in the network with fixed probabilities depending only on the source and target queues, and where the exogenous inputs are independent Poisson processes, the state distribution of each queue is that of an ordinary $M/M/s$ queue without feedback but with traffic parameters depending on the feedback probabilities. (This is a result of Jackson [1].) The point is that, in spite of the presence of feedback, the queues behave, in certain respects, as if the total input processes were Poisson. In this connection the present author wrote [2], ". . . The combined input to a queue, new arrivals and returning customers, is apparently not Poisson." This conjecture was supported only by a heuristic argument, which was not altogether convincing to some readers. Hence, it was thought worthwhile to provide a rigorous proof that the input is not Poisson in the simplest case of a Jackson-type network, namely, a single $M/M/1$ queue with feedback. The proof takes the form of a calculation of the marginal distribution of an interarrival interval of the input of such a queue, and it provides as a byproduct the explicit form of this distribution.

Before proceeding with the calculation it is well to make the point explicitly that fed-back calls, and hence also the total input stream, see the same state distribution as the exogenous calls. That is, we assert the proposition that the state distribution at arbitrary arrival instants, fed-back or exogenous, is the same as that at the exogenous arrival instants alone. To establish this proposition we recall first that in any equilibrium queuing system with individual arrival and service, the departing (not fed-back) calls leave the same state distribution as seen by exogenous calls [3]. Next we observe that the fed-back calls leave the same state distribution as the departing calls since calls which are fed back (or which leave) are selected with fixed probabilities (thus independently of the state). Since fed-back calls see the same state that they leave, the proposition follows.

In addition to p, the feedback probability, let

- λ original arrival rate
- μ service rate
- π_i equilibrium probability of state i at an arbitrary instant
- q $= 1 - p$.

It is known [1] that, if $\lambda/\mu q < 1$,

$$\pi_i = [1 - (\lambda/\mu q)](\lambda/\mu q)^i, \tag{1}$$

Paper approved by the Associate Editor for Computer Communication of the IEEE Communications Society for publication without oral presentation. Manuscript received November 20, 1975; revised January 6, 1976.

The author is with Bell Laboratories, Holmdel, NJ 07733.

and it is also known that π_i is the probability of state i at (just before) an exogenous arrival, because of the Poisson character of this arrival process. Therefore, by the proposition proved above, π_i is the state probability at an arbitrary arrival instant in the total arrival process.

Let $F(t)$ be the complementary distribution function of the length of an interarrival interval, i.e., the probability that in the total input stream the first arrival epoch after an arbitrary arrival epoch t_0 will not be sooner than $t_0 + t$; and let $G(t)$ be the same for the first fed-back call of the calls in the system at t_0. Clearly $F(t) = e^{-\lambda t}G(t)$, and it remains only to calculate $G(t)$. To do this, consider the density function of the length of the interval between t_0 and the arrival instant, $t_0 + t$, of the first fed-back call,

$$g(t) = -\frac{dG(t)}{dt} = \sum_{i=0}^{\infty} \pi_i g_i(t), \tag{2}$$

where $g_i(t)$ is the conditional density given that the state at t_0 is i.

Now

$$g_i(t) = \sum_{j=1}^{i+1} q^{j-1} p\mu e^{-\mu t}(\mu t)^{j-1}/(j-1)! \tag{3}$$

where the jth term in the sum is the probability that the jth call is the first call fed back multiplied by the conditional density of the service-completion epoch of the jth call (the convolution of j exponentials each with parameter μ).

Combining (1) and (2) and using the value of π_i,

$$g(t) = [1 - (\lambda/q\mu)]\, p\mu e^{-\mu t} \sum_{i=0}^{\infty} (\lambda/q\mu)^i \cdot \sum_{j=1}^{i+1} (q\mu t)^{j-1}/(j-1)!,$$

and reversing the order of summation,

$$g(t) = [1 - (\lambda/q\mu)]\, p\mu e^{-\mu t} \cdot \sum_{j=1}^{\infty} (q\mu t)^{j-1}/(j-1)! \sum_{i=j-1}^{\infty} (\lambda/q\mu)^i = p\mu e^{-(\mu-\lambda)t}.$$

Then integrating, and taking into account the probability that no call present at t_0 is fed back,

$$G(t) = 1 - p\mu(1 - e^{-(\mu-\lambda)t})/(\mu - \lambda)$$

or

$$F(t) = [(q\mu - \lambda)e^{-\lambda t} + p\mu e^{-\mu t}]/(\mu - \lambda).$$

Since $F(t)$ is a mixture of two exponentials, the total input process is not Poisson.

REFERENCES

[1] J. R. Jackson, "Networks of waiting lines," *Operations Res.*, vol. 5, pp. 518-521, 1957.
[2] P. J. Burke, "Output processes and tandem queues," in *Proc. Symp. Computer-Communications Networks and Teletraffic.* New York: Polytech. Inst. of Brooklyn Press, 1972, pp. 419-428.
[3] R. B. Cooper, *Introduction to Queuing Theory.* New York: Macmillan, 1972, p. 154.

Correction to "A Per-Channel A/D Converter Having 15-Segment μ-255 Companding"

JAMES C. CANDY

On page 42 of the above paper,[1] the picture designated William H. Ninke more nearly resembles me than either of the other two pictures. In pointing out this fact, I imply no criticism of the picture assigned to me, although it has the appearance of Dr. Ninke. Contrary to expectations, Dr. Wooley is pleased with his picture.

Paper approved by the Associate Editor for Data Communication Systems of the IEEE Communications Society for publication. Manuscript received January 26, 1976.

The author is with Bell Laboratories, Holmdel, NJ 07733.

[1] J. C. Candy, W. H. Ninke, and B. A. Wooley, *IEEE Trans. Commun.*, vol. COM-24, pp. 33-42, Jan. 1976.