Some New Similarity Measures for Histograms

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Abstract

Similarity measures, originally introduced to express the degree of comparison between two fuzzy sets, can be applied in several ways to digital images. In [1–3] we have shown how similarity measures were used for the construction of image quality evaluation measures. In this paper we illustrate how fuzzy similarity measures can be useful for the comparison of histograms of digital images. We will show that similarity measures can be applied to two different types of histograms: normalized histograms and ordered normalized histograms.

1. Introduction

An important problem in image processing is the comparison of images: if different algorithms are applied to an image, we need an objective measure to compare the output images. It is well-known that classical measures, such as the $RMSE$ (root mean square error), do not always give convincing results. Furhtermore, measures of comparison between images are an indispensable tool in image retrieval. In this paper we focus on a low-level image comparison, namely histogram-based comparison of images. Traditionally, one uses a standard distance measure to calculate the distance between two histograms. In this paper, we investigate the possible application of fuzzy similarity measures to histograms of digital images.

After some preliminaries in Section 2 we briefly discuss the relevant properties we impose to a similarity measure in order to be applicable to digital images. Then, we give a short overview of classical methods for histogram comparison in Section 4. After this, the first way in which similarity measures, originally introduced to express the degree of comparison between two fuzzy sets, can be applied to normalized histograms of digital images is outlined in Section 5. However, similarity measures can be applied to a second type of histograms, namely ordered and normalized histograms. This is explained in Section 6. Finally, we illustrate the behaviour of the appropriate similarity measures with some examples in Section 8.

2. Preliminaries

In this section we repeat some basic notions of fuzzy set theory and similarity measures.

2.1. Fuzzy sets

A fuzzy set [7] A in a universe X is characterised by a $X - [0,1]$ mapping μ_A, which associates with every element x in X a degree of membership $\mu_A(x)$ of x in the fuzzy set A. In the following, we will denote the degree of membership by $A(x)$, and the class of fuzzy sets in a universe X by $\mathcal{F}(X)$.

2.2. Similarity measures

In the literature a lot of measures are proposed to express the similarity or equality between fuzzy sets. There is no unique definition, but the most frequently used is the following [8]. A *similarity measure* is a fuzzy binary relation in $\mathcal{F}(X)$, i.e. a $\mathcal{F}(X) \times \mathcal{F}(X) \to [0,1]$ mapping that is reflexive, symmetric and min-transitive. However, not every measure in the literature satisfies this definition. Therefore, a similarity measure will here be understood as a measure we can use to compare fuzzy sets, or objects which can be identified with fuzzy sets.

2.3. Fuzzy logical operators

Several classes of similarity measures are based on fuzzy logical operators, such as conjunctors, disjunctors and implicators.

A conjunctor is an increasing $[0,1]^2 - [0,1]$ mapping $\mathcal{T}$ satisfying the border conditions $\mathcal{T}(0,0) = \mathcal{T}(0,1) = \mathcal{T}(1,0) = 0$ and $\mathcal{T}(1,1) = 1$. The most popular conjunctors are T_M, T_P and T_W: $T_M(x,y) = \min(x,y)$, $T_P(x,y) = x \cdot y$ and $T_W(x,y) = \max(0, x+y-1)$, for all (x,y) in $[0,1]^2$.

A disjunctor is a increasing $[0,1]^2 - [0,1]$ mapping $\mathcal{S}$ satisfying the border conditions $\mathcal{S}(1,0) = \mathcal{S}(0,1) = \mathcal{S}(1,1) = 1$ and $\mathcal{S}(0,0) = 0$. The most popular disjunctors are S_M, S_P and S_W: $S_M(x,y) = \max(x,y)$, $S_P(x,y) = x + y - x \cdot y$ and $S_W(x,y) = \min(1, x+y)$, for all (x,y) in $[0,1]^2$.

An implicator is a $[0,1]^2 - [0,1]$ mapping $\mathcal{I}$ with decreasing first and increasing second partial maps, which satisfies the border conditions $\mathcal{I}(0,0) = \mathcal{I}(0,1) = \mathcal{I}(1,1) = 1$ and $\mathcal{I}(1,0) = 0$. The most popular implicators are I_{KD}, I_W and I_R: $I_{KD}(x,y) = \max(1-x, y)$, $I_W(x,y) = \min(1, 1-x+y)$ and $I_R(x,y) = 1 - x + x \cdot y$, for all (x,y) in $[0,1]^2$.

3. Relevant properties for image processing

In order to investigate whether similarity measures can be applied for image comparison, we evaluated about 50 measures with respect to several relevant properties. We have considered the following properties [3]:

Reflexivity: for two identical images one may expect that the similarity measure has output 1.

Symmetry: the output of the similarity measure is expected to be independent of the order in which the two input images are considered.

Reaction to noise (e.g. salt & pepper noise or gaussian noise): a good similarity measure should not be affected too much due to noise (since a noisy image is coming from an original one, it has to be similar to the original image), and should be decreasing with respect to an increasing noise percentage.

Reaction to enlightening and darkening: if one enlightens or darkens an image with a constant value, the similarity measure should return a high value (indeed, one considers almost identical images). One also expects a decreasing behaviour with respect to an increasing enlightening or darkening percentage.

This list should not be considered as complete: depending on the application or the type of images that have to be compared, some properties will be less relevant and sometimes completely different properties will have to be investigated.

4. Classical measures of comparison for histograms

The histogram of a greyscale image is a chart that shows the frequency distribution of the different grey levels. The value of the histogram of a greyscale image A in a grey level g equals the total amount of pixels in the image A with grey level g, and will be denoted as $h_A(g)$. In this way, we obtain the following expression for the value of the histogram of the image A in the grey level g:

$$h_A(g) = \sum_{(i,j)\in X} \delta(g - A(i,j)),$$

with δ the Dirac function and X the universe of image points.

Usually, one uses some kind of distance function to determine how much two histograms differ from each other. Since histograms are often understood as vectors, the most popular metrics for histograms are induced by the Minkowski norm (L_p) which is used in vector spaces. Two popular examples are the *Manhattan distance* and the *Euclidean distance* induced by respectively the L_1-norm and the L_2-norm, and are given by the following expressions:

$$\begin{aligned} d_{L_1}(h_A, h_B) &= \sum_{g=0}^{|G|-1} |h_A(g) - h_B(g)| \\ d_{L_2}(h_A, h_B) &= \sqrt{\sum_{g=0}^{|G|-1} |h_A(g) - h_B(g)|^2}, \end{aligned}$$

with G the universe of grey levels.

Another technique for the comparison of histograms was introduced by Swain and Ballard [6] and is called the *histogram intersection*. For two histograms h_A and h_B, the histogram intersection is defined as

$$d_I(h_A, h_B) = \sum_{g\in G} \min(h_A(g), h_B(g)).$$

However, if A and B have the same size, one can proove that the sum of $d_{L_1}(h_A, h_B)$ and $2d_I(h_A, h_B)$ equals a constant. Consequently, there is no significant difference between the histogram intersection and the Manhattan distance.

Another distance function for histograms is based on statistics: the χ^2-test [4] defined as

$$d_{\chi^2}(h_A, h_B) = \sum_{g\in G} \frac{(h_A(g) - h_B(g))^2}{(h_A(g) + h_B(g))}.$$

There also exist techniques for histogram comparison based on cumulative histograms, which are defined as follows

$$\tilde{h}_A(g) = \sum_{g' \leq g} h_A(g').$$

The following metrics [5] make use of cumulative histograms:

$$\begin{aligned} \tilde{d}_{L_1}(h_A, h_B) &= \sum_{g\in G} |\tilde{h}_A(g) - \tilde{h}_B(g)| \\ \tilde{d}_{L_2}(h_A, h_B) &= \sqrt{\sum_{g\in G} \left(\tilde{h}_A(g) - \tilde{h}_B(g)\right)^2}. \end{aligned}$$

5. Direct application of similarity measures to histograms

First of all, it is meaningful to compare two histograms in the framework of fuzzy set theory, because the histogram of an image can be transformed to a fuzzy set in the universe of grey levels by dividing the values of the histogram in every grey leval by the maximum amount of pixels with the same grey value. In this way the most typical grey value gets membership degree 1 in the fuzzy set associated with the histogram and every other less typical grey value gets a smaller membership degree. Consequently, a normalized histogram is in accordance with the intuitive idea behind a fuzzy set: the most typical element in the universe gets membership degree 1 and all other less typical elements belong to the fuzzy set to a less extent which can be expressed by membership degrees smaller than 1. In this way we obtain the following expression for the membership degree of the grey value g in de fuzzy set Fh_A associated with the histogram h_A of the image A:

$$Fh_A(g) = \frac{h_A(g)}{h_A(g_M)}$$

with $h_A(g_M) = \max_{g \in G} h_A(g)$. As histograms of digital images can be identified with fuzzy sets in the universe of grey values, it is interesting to investigate whether similarity measures, originally introduced to express the degree of comparison between two fuzzy sets, can be applied to normalized histograms in a meaningful way. In this way we compare two images on a histogram-level, and the frequencies of the different grey values are compared, grey value per grey value.

6. Application of similarity measures to ordered histograms

Similarity measures can be applied in a second way to associated histograms of digital images. The values of a histogram can be ordered in such a way the least occuring grey value is placed in the first position of the histogram and the rest of frequencies are ordered in increasing order. Again, the histogram is normalized analogously to the first case, and consequently the most typical grey value gets membership degree 1 in the fuzzy set associated with the histogram and all the other membership degrees are all smaller than or equal to 1 and are ordered in increasing order. Again, we can apply the different similarity measures to these ordered and normalized histograms. In contrast with the first application of similarity measures to normalized histograms, where the frequencies of the different grey levels are compared grey level per grey level, in this case the frequency of the most occuring grey level in the image A is compared with the most occuring grey level in the image B, the frequency of the second most occuring grey level in the image A is compared with the second most occuring grey level in the image B, ... The frequencies of the different grey values are compared frequency per frequency, with respect to an increasing order of the different frequencies. So, it can happen that two frequencies of two different grey values are compared to each other, depending on the place they take in the ordered histogram. If the ordered histogram of an image A is denoted as o_A, we obtain the following expression for the fuzzy set associated with the ordered histogram of the image A. For $i = 1, ..., |G|$, with G the universe of grey levels:

$$Oh_A(i) = \frac{o_A(i)}{o_A(|G|)},$$

with $o_A(g) = \max_{g \in G} h_A(g)$.

7. Appropriate measures of comparison for histograms

A profound experimental study of the applicability of similarity measures to normalized histograms resulted in 15 similarity measures which are appropriate for the comparison of images, i.e. they satisfy the list of relevant properties we impose to a similarity measure in order to be applicable in image processing. We recollect shortly the expressions of several appropriate similarity measures:

$$\begin{aligned}
H_1(A,B) &= 1 - \left(\frac{1}{L} \sum_{g \in G} |Fh_A(g) - Fh_B(g)|^r \right)^{\frac{1}{r}}, \\
&\quad \text{with } r \in \mathbb{N}\backslash\{0\} \\
H_3(A,B) &= 1 - \frac{\sum_{g \in G} |Fh_A(g) - Fh_B(g)|}{\sum_{g \in G} (Fh_A(g) + Fh_B(g))} \\
H_6(A,B) &= \frac{|Fh_A \cap Fh_B|}{|Fh_A \cup Fh_B|} \\
H_9(A,B) &= \frac{\min(|Fh_A|, |Fh_B|)}{|Fh_A \cup Fh_B|} \\
H_{12}(A,B) &= \frac{|(Fh_A \triangle Fh_B)^c|}{\max(|(Fh_B \backslash Fh_A)^c|, |(Fh_A \backslash Fh_B)^c|)} \\
H_{20}(A,B) &= \frac{1}{MN} \sum_{g \in G} \frac{\min(Fh_A(g), Fh_B(g)}{\max(Fh_A(g), Fh_B(g)}.
\end{aligned}$$

with $L = |G|$ the total amount of different grey levels and G the universe of grey levels. Furthermore we use the minimum T_M to model the intersection between two fuzzy sets, the maximum S_M to model the union between two fuzzy sets, and the standard negator N_s to model the complement of a fuzzy set. The symmetric difference between two fuzzy sets A and B in a universe X is defined as $A \Delta B = (A \backslash B) \cup (B \backslash A)$, with $A \backslash B = A \cap B^c$. The cardinality of a fuzzy set A (with finite support) in a universe

X is defined as: $|A| = \sum_{x \in X} A(x)$.

If the similarity measures are applied to ordered histograms we obtain 22 similarity measures which satisfy the list of relevant properties. Besides the 15 similarity measures which were appropriate for direct application, we found 7 extra similarity measures which are appropriate for application to ordered histograms. Also in this case we recollect the expressions of the several appropriate similarity measures:

$$
\begin{aligned}
OH_1(A,B) &= 1 - \left(\frac{1}{L} \sum_{i=1}^{|G|} |Oh_A(i) - Oh_B(i)|^r \right)^{\frac{1}{r}}, \\
&\quad \text{with } r \in \mathbb{N} \backslash \{0\} \\
OH_2(A,B) &= 1 - \max_{i \in G} |Oh_A(i) - Oh_B(i)| \\
OH_3(A,B) &= 1 - \frac{\sum_{i=1}^{|G|} |Oh_A(i) - Oh_B(i)|}{\sum_{i=1}^{|G|} (Oh_A(i) + Oh_B(i))} \\
OH_5(A,B) &= \frac{\min(|Oh_A|, |Oh_B|)}{\max(|Oh_A|, |Oh_B|)} \\
OH_6(A,B) &= \frac{|Oh_A \cap Oh_B|}{|Oh_A \cup Oh_B|} \\
OH_9(A,B) &= \frac{\min(|Oh_A|, |Oh_B|)}{|Oh_A \cup Oh_B|} \\
OH_{11}(A,B) &= \frac{\min(|Oh_A \backslash Oh_B|, |Oh_B \backslash Oh_A|)}{\max(|Oh_A \backslash Oh_B|, |Oh_B \backslash Oh_A|)} \\
OH_{12}(A,B) &= \frac{|(Oh_A \triangle Oh_B)^c|}{\max(|(Oh_B \backslash Oh_A)^c|, |(Oh_A \backslash Oh_B)^c|)} \\
OH_{20}(A,B) &= \frac{1}{MN} \cdot \sum_{i=1}^{|G|} \left[\frac{\min(Oh_A(i), Oh_B(i))}{\max(Oh_A(i), Oh_B(i))} \right].
\end{aligned}
$$

8. Some examples

The first experiment is an illustration of how the similarity measures react to salt & pepper noise and gaussian noise. We will add three different percentages salt & pepper noise and three different amounts of gaussian noise to the "cameraman" image. The original image and the noisy images are displayed in Figure 1. The results are shown in Table 1. One can verify that the values are relatively high, and that the similarity values slightly decrease with respect to an increasing noise level. In the second experiment we apply the different measures to an enlightened version of the "cameraman" image (see Figure 1). The results of this experiment are also displayed in Table 1. In this case the new measures for histogram comparison clearly outperform the classical measures of comparison, because the results of the classical measures are far too high, in comparison with the results of the experiment with an increasing noise percentage. Finally, we apply the different similarity measures to the histograms of totally different images, more precisely to the "cameraman" image and the "Lena" image (see Figure 1). In this case we obtain the lowest value for each of the new similarity measures in comparison with the results of the other experiments, which is of course a satisfactory result.

9. Conclusion

In this paper we have illustrated how similarity measures, originally introduced to express the degree of comparison between two fuzzy sets, can be applied to histograms of images. In the first place, the similarity measures were applied directly to the normalized histograms of the considered images, and secondly the similarity measures were applied to normalized ordered histograms. We gave a thorough overview of the similarity measures which can be applied succesfully to the histograms of digital images. All the appropriate similarity measures were illustrated with some examples.

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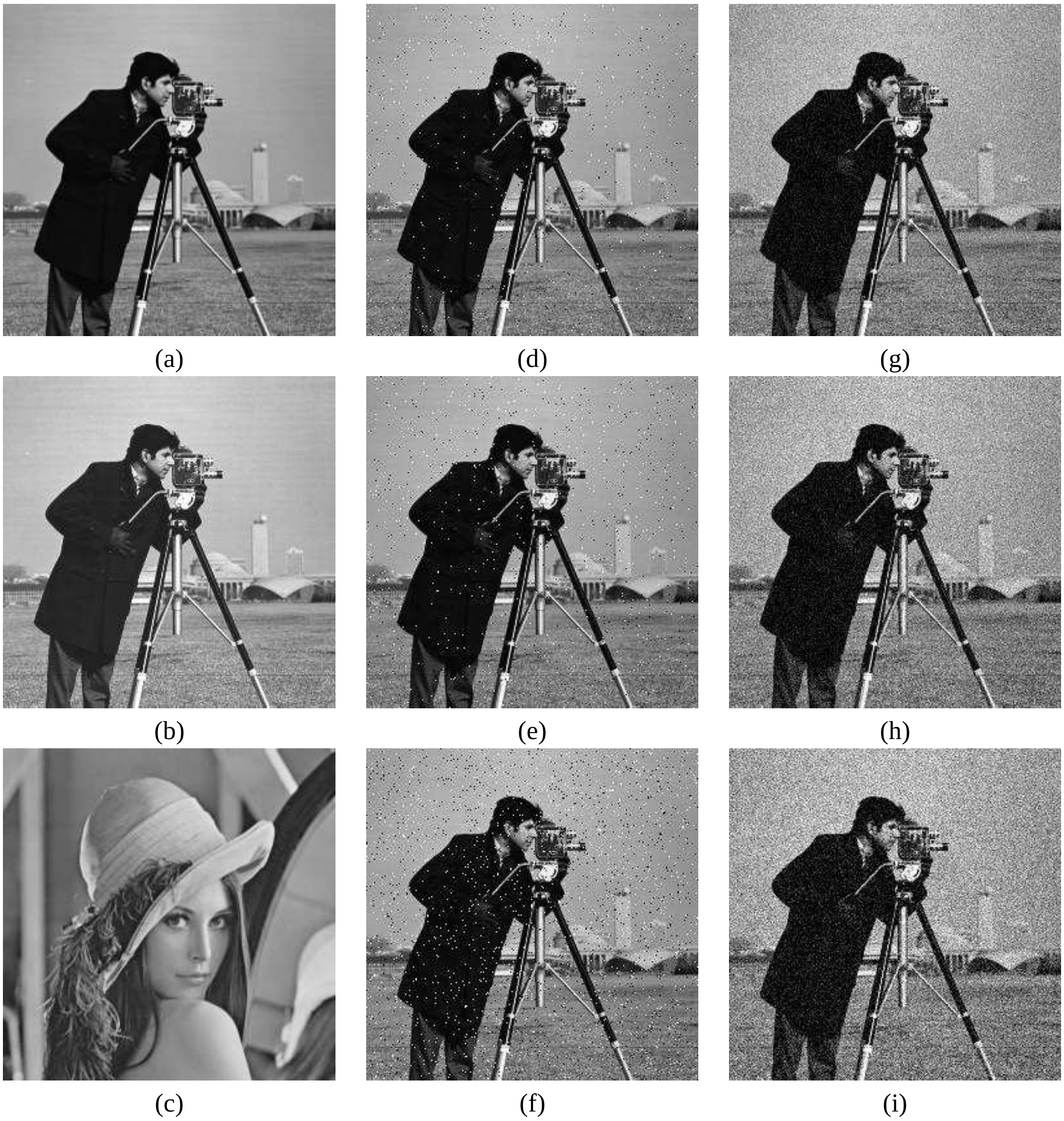

Figure 1: (a) The original "cameraman" image, (b) enlightened version of the "cameraman" image, (c) the "lena" image (d) the "cameraman" image corrupted with 10% salt & pepper noise, (e) the "cameraman" image corrupted with 15% salt & pepper noise, (f) the "cameraman" image corrupted with 40% salt & pepper noise, (g) the "cameraman" image corrupted with gaussian noise ($\sigma = 14$), (h) the "cameraman" image corrupted with gaussian noise ($\sigma = 18$), (i) the "cameraman" image corrupted with gaussian noise ($\sigma = 25$).

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
$H_1(r=1)$	0.99788	0.99676	0.99258	0.92379	0.90553	0.89085	0.84054	0.70588
$H_1(r=2)$	0.98315	0.97299	0.92946	0.83796	0.81253	0.79232	0.72676	0.62255
$H_1(r=4)$	0.9434	0.90913	0.76282	0.67235	0.6468	0.62851	0.56585	0.52299
H_3	0.99305	0.98943	0.97603	0.70358	0.59283	0.48903	0.49452	0.42781
H_4	0.99971	0.99926	0.99507	0.97416	0.96533	0.9573	0.92597	0.85927
H_6	0.98619	0.97908	0.95318	0.54271	0.4213	0.32365	0.32848	0.27211
H_{6c}	0.9975	0.99618	0.99126	0.91621	0.89855	0.88483	0.82705	0.66955
H_7	0.98745	0.98179	0.95836	0.59532	0.4527	0.34379	0.49452	0.30365
H_{7c}	0.99773	0.99669	0.99226	0.93129	0.90961	0.89375	0.90534	0.70269
H_9	0.98747	0.98185	0.95859	0.63109	0.49065	0.38224	0.66424	0.376
H_{9c}	0.99773	0.99669	0.99227	0.9324	0.91071	0.89481	0.91352	0.71671
H_{12}	0.9978	0.9968	0.99252	0.93138	0.90964	0.89377	0.90537	0.70929
H_{I_3}	0.98294	0.9742	0.94265	0.53963	0.42	0.32286	0.32770	0.23567
$H_{I_{3c}}$	0.99758	0.99631	0.99155	0.91631	0.89859	0.88486	0.82710	0.67655
H_{18c}	0.99788	0.99676	0.99258	0.92379	0.90553	0.89085	0.84054	0.70588
H_{20}	0.98336	0.98005	0.97151	0.6417	0.56439	0.50757	0.45568	0.27719
H_{20c}	0.99743	0.99605	0.99173	0.90768	0.89187	0.88029	0.82433	0.6566
$OH_1(r=1)$	0.99809	0.99734	0.99414	0.93846	0.92058	0.90424	0.99884	0.78719
$OH_1(r=2)$	0.99657	0.99518	0.98604	0.87635	0.84416	0.81887	0.99768	0.74345
$OH_1(r=4)$	0.99363	0.99112	0.96883	0.77034	0.72603	0.69296	0.99606	0.70846
OH_2	0.98187	0.97211	0.89902	0.3962	0.30065	0.23089	0.98891	0.57636
OH_3	0.99376	0.99132	0.98107	0.76064	0.65773	0.55171	0.99632	0.58599
OH_4	0.99999	0.99998	0.99981	0.98509	0.97619	0.96765	0.99999	0.93513
OH_5	0.98872	0.98457	0.9638	0.69228	0.52722	0.40602	1.00000	0.41958
OH_{5c}	0.99796	0.99719	0.99327	0.94775	0.92192	0.90383	1.00000	0.75219
OH_6	0.9876	0.98279	0.96284	0.61373	0.49001	0.38094	0.99267	0.41441
OH_{6c}	0.99775	0.99687	0.99309	0.93179	0.91402	0.89824	0.99862	0.74945
OH_7	0.98816	0.98367	0.96331	0.6436	0.50225	0.38786	0.99632	0.41593
OH_{7c}	0.99785	0.99703	0.99318	0.93949	0.91779	0.90088	0.99931	0.75062
OH_9	0.98817	0.98369	0.96333	0.66015	0.51438	0.39878	0.99634	0.41806
OH_{9c}	0.99785	0.99703	0.99318	0.93998	0.91813	0.90117	0.99931	0.75101
OH_{11}	0.98607	0.98096	0.95487	0.67858	0.51139	0.38938	1.00000	0.31096
OH_{11c}	0.99803	0.99729	0.99351	0.94813	0.92233	0.90425	1.00000	0.76781
OH_{12}	0.99793	0.99713	0.99342	0.93992	0.91823	0.90132	0.99934	0.76635
OH_{16e}	0.97805	0.96668	0.89258	0.38055	0.29574	0.22719	0.98099	0.56986
OH_{16h}	0.97798	0.96653	0.89185	0.35671	0.28431	0.21487	0.98090	0.56508
OH_{I_3}	0.98468	0.97877	0.95368	0.59737	0.47336	0.36392	0.99082	0.3053
$OH_{I_{3c}}$	0.99783	0.99697	0.99333	0.93227	0.91448	0.89869	0.99867	0.76523
OH_{18c}	0.99809	0.99734	0.99414	0.93846	0.92058	0.90424	0.99884	0.78719
OH_{20}	0.9522	0.95165	0.9473	0.68173	0.64546	0.55743	0.88798	0.31514
OH_{20c}	0.99752	0.99627	0.98846	0.92031	0.90526	0.8924	0.99788	0.70712
d_{L_1}	1268	2024	5182	28138	32886	38946	66254	57866
d_{L_2}	453.81	723.81	1851.25	4186.20	4920.87	5810.41	7095.46	5631.92
d_I	64902	64524	62945	51467	49093	46063	32409	36603
d_{χ^2}	638.18	1022.18	2648.91	13613.91	17062.46	21534.64	48238.22	37010.11

Table 1: Results of the classical measures of comparison for histograms and the appropriate fuzzy similarity measures for the comparison of histograms: (a) 10% salt & pepper noise, (b) 15% salt & pepper noise, (c) 40% salt & pepper noise, (d) gaussian noise with $\sigma = 14$, (e) gaussian noise with $\sigma = 18$, (f) gaussian noise with $\sigma = 25$, (g) enlightened version, (h) “cameraman” image vs. “lena” image.