CS206 Quiz No. #2

Date: March 28, 2006
Time: 1 hour

- This is an open book/notes/material-brought-to-class exam.
- Be brief and stick to the point that has been asked.
- If you absolutely need to make any assumptions, state them clearly. If the assumptions are unreasonable, no marks will be awarded for the part of the solution using the assumptions.
- **Do not copy solutions or indulge in unfair means.**

1. *[5+ 5 + 5 marks]* Given a predicate logic formula ϕ, define the *alternation depth* of ϕ as the minimum number of changes of quantifiers (from existential to universal or vice versa) in the prefix if we write ϕ in a prenex normal form. For example, the number of changes of quantifiers in the prefix of $\forall x \exists y \exists z \forall w \exists v P(x, y, z, w, v)$ is 3.

   Note that a formula may have multiple prenex normal forms; the alternation depth of ϕ is the minimum number of quantifier changes in the prefix among all such prenex normal forms of ϕ.

   Let $\phi(z) = \forall x \exists y (((\forall x P(x, y, z)) \rightarrow (\exists y P(x, y, z))) \rightarrow (\exists x \forall y P(x, y, z)))$, where $P(x, y, z)$ is a ternary predicate.

   (a) Give a prenex normal form for ϕ in which the number of changes of quantifiers in the prefix is **minimized** and indicate the alternation depth of ϕ.

   (b) Is it possible to write a Skolem normal form for $\exists z \phi(z)$ in which all Skolem functions are of arity one? If so, give the corresponding Skolem normal form. Else, give justification for your answer.

   (c) In the formula $\phi(z)$ above, suppose every instance of $P(x, y, z)$ is replaced by a predicate logic formula $\psi(x, y, z)$, where the alternation depth of ψ is k. Give as tight an upper bound as you can of the alternation depth of $\phi(z)$ in terms of k. You must provide justification for your answer to score marks.

2. *[10 marks]* Show using the Compactness Theorem that it is not possible to write a predicate logic sentence ϕ using only the equality predicate and a binary predicate E (no other function symbols are allowed), such that (i) all models of ϕ are directed graphs containing at least one cycle (including self loops), and (ii) any directed graph containing at least one cycle (including self loops) gives rise to a model of ϕ.