CS206 Homework #2

Max marks: 85 **Due March 31, 2008**

- *Be brief, complete and stick to what has been asked.*
- ***Do not copy solutions from others.***

1. *[15 marks]* Let $\phi_{K,N}$ be a predicate logic formula with no free variables (such formulae are called *sentences*) and with only one predicate symbol P in addition to the *equality* predicate. Suppose further that P is a unary predicate. A model $\mathcal{M}$ for $\phi_{K,N}$ consists of a set of elements, called the domain (or universe) $D_{\mathcal{M}}$, and an interpretation $P_{\mathcal{M}} : D_{\mathcal{M}} \rightarrow \{\mathsf{true}, \mathsf{false}\}$ of the predicate P. Therefore a model $\mathcal{M}$ for $\phi_{K,N}$ can be represented as a pair $(D_{\mathcal{M}}, P_{\mathcal{M}})$. You are told the following additional facts about $\phi_{K,N}$ and its models.

   - In every model $\mathcal{M}$ such that $\mathcal{M} \models \phi_{K,N}$, the domain $D_{\mathcal{M}}$ contains *at most* K distinct elements, and the interpretation $P_{\mathcal{M}}$ of P evaluates to true on *at least* N distinct elements of $D_{\mathcal{M}}$.
   - Any model $\mathcal{M}$ in which $D_{\mathcal{M}}$ contains *at most* K distinct elements and the interpretation $P_{\mathcal{M}}$ of P evaluates to true on *at least* N distinct elements of $D_{\mathcal{M}}$, satisfies $\mathcal{M} \models \phi_{K,N}$.

   (a) *[5 marks]* Give a predicate logic formula $\phi_{K,N}$ satisfying the above conditions. You may use the notation $\exists x_1 \exists x_2 \ldots \exists x_r$ or $\forall y_1 \forall y_2 ... \forall y_r$ to denote a sequence of r quantifications of the same type ($\exists$ or $\forall$).

   (b) *[5 marks]* Show using natural deduction that $\phi_{2,3} \vdash \bot$.

   (c) *[5 marks]* Show using natural deduction that $\phi_{3,3} \vdash \forall x\, P(x)$.

   In the last two subquestions, $\phi_{2,3}$ and $\phi_{3,3}$ must be obtained by substituting appropriate values for K and N in your answer to the first subquestion.

2. *[25 marks]* In this question, we wish to express properties of directed graphs using predicate logic.

   Consider a predicate logic formula ϕ containing a single binary predicate E in addition to the equality predicate. A model $\mathcal{M}$ for ϕ consists of a domain (or universe) $D_{\mathcal{M}}$ and an interpretation $E_{\mathcal{M}} : D_{\mathcal{M}} \times D_{\mathcal{M}} \rightarrow \{\mathsf{true}, \mathsf{false}\}$ of the predicate E. Such a model $\mathcal{M}$ can also be viewed as a directed graph $G_{\mathcal{M}}$, where the elements of $D_{\mathcal{M}}$ are the vertices of the graph, and a directed edge exists from a to b if and only if $E_{\mathcal{M}}(a, b)$ is true. For purposes of this question, we will assume that $D_{\mathcal{M}}$ is finite.

   In each of the following cases, you are required to give a predicate logic sentence (formula without free variables) ϕ such that $\mathcal{M} \models \phi$ if and only if

   (a) *[5 marks]* $G_{\mathcal{M}}$ is a tree **of at most** 4 **nodes**, with a unique root.

   (b) *[5 marks]* $G_{\mathcal{M}}$ is a forest **where each tree in the forest has at most** 4 **nodes**. Note that a forest is a collection of one or more trees, each with a unique root.

   (c) *[5 marks]* $G_{\mathcal{M}}$ is a collection of one or more simple cycles. Note that a simple cycle is a graph in which (i) it is possible to start from any node and follow directed edges to return to the same node, and (ii) the *only way* to do the above (without visiting the starting node in between) is by visiting all other nodes on the simple cycle exactly once.

(d) *[5 marks]* All cycles, if any, in $G_{\mathcal{M}}$ contain at least 3 vertices.

(e) *[5 marks]* There is at least one infinite path (containing repeated vertices of course) starting from every vertex in $G_{\mathcal{M}}$.

You must provide brief justification (3-5 lines) for each of your answers.

3. *[25 marks]* In this question, we wish to state certain properties of natural numbers in predicate logic. You may use the predicates $<$ and $=$, and the functions $*$ and $+$ on natural numbers with the usual interpretation. You may also use $one()$ as a nullary function that returns the value 1. **You are not allowed to use any other predicate or function symbols.**

   Give predicate logic sentences expressing the following properties:

   (a) *[10 marks]* There are natural numbers that cannot be expressed as one natural number raised to the power of another natural number distinct from 1.

   (b) *[5 marks]* There are natural numbers that cannot be expressed as the product of distinct natural numbers, none of which is 1.

   (c) *[10 marks]* There are infinitely many natural numbers that have only one way of factorizing them as the product of two natural numbers.

4. *[10 marks]* Give proofs in natural deduction (within the number of steps specified) for each of the following sequents. Each step should be either a premise or an application of a natural deduction rule. Also indicate which rule you are applying at each step.

   (a) *[5 marks]* $\exists y \forall x \neg(f(x) = y) \;\vdash\; \exists x \exists y \neg(x = y)$ *(within 10 steps)*

   (b) *[5 marks]* $\forall x \neg(f(x) = x),\; \forall x \forall y \forall z\, (x = y) \vee (y = z) \vee (x = z) \;\vdash\; \exists x \exists y\, (f(x) = y) \wedge (f(y) = x)$ *(within 15 steps)*

5. *[10 marks]* Once upon a time, there was a logician who, by some strange stroke of fate, ended up ruling a land (such things aren't common in recent times!) Being a logician, the ruler was interested in finding out whether the prime minister, who was entrusted with key responsibilities, was logically consistent in thinking. So one evening, the prime minister (PM) was summoned and was asked to respond in "Yes/No" to questions that the ruler (R) would pose.

   The following short question-answer session ensued:

   R: Is there a happy person in my empire who knows somebody who in turn knows an unhappy person?

   PM: No

   R: Is there a happy person in my empire who is not known to even one other happy person?

   PM: No

   R: Is there a happy person in my empire who knows another person who is unhappy?

   PM: Yes

   At this point, the logician-turned-ruler remarked that the prime minister is being logically inconsistent and recommended that a crash course in predicate logic be given to the prime minister. As the first assignment, the minister was asked to use natural deduction to prove that the sequence of answers given above is logically inconsistent. We must help the minister in this noble effort.

   We will use a unary predicate $H(x)$ that evaluates to `true` iff x is happy, and a binary predicate $K(x, y)$ that evaluates to `true` iff x knows y, in addition to the usual equality predicate (if needed). No other predicates or functions must be used. Note that K is not necessarily a reflexive, symmetric or transitive relation, and we must not make any such assumptions.

(a) *[2 + 2 + 2 marks]* Express the information provied by each of the above question-answer pairs as a formula in predicate logic. Thus, you should obtain three formulae ϕ_1, ϕ_2, ϕ_3 using the predicates H and K (possibly in addition to the equality predicate), and using no other predicates or functions.

(b) *[4 marks]* Show using natural deduction that $\phi_1, \phi_2, \phi_3 \vdash \bot$.