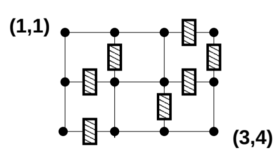
Time: 60 mins | Total marks: 20

- ***Please write your Roll No. on the top right of each sheet.***
- ***You must write your answers only in the spaces provided.***
- ***The exam is open book and notes.***
- ***Results/proofs covered in class/problem sessions/assignments may simply be cited, unless specifically asked for.***
- ***Unnecessarily lengthy solutions will be penalized.***
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others or indulge in unfair means.***

1. A 2−dimensional maze is defined by an $n \times m$ grid (n not necessarily equal to m), plus four $n \times m$ *blocking matrices* named Top, Bottom, Left and Right. The grid points are indexed from $(1, 1)$ (top left corner) to (n, m) (bottom right corner). Each grid point is potentially connected to its neighbouring grid points by paths in the top, bottom, left and right directions. Some of these paths may however be blocked. The information about blocked paths is given by the four blocking matrices. Each element of a blocking matrix is either 0 or 1. The path to the left of grid point (i, j) is blocked or does not exist iff $\mathsf{Left}[i, j] = 1$. Similarly, the path to the right of grid point (i, j) is blocked (or non-existent) iff $\mathsf{Right}[i, j] = 1$, the path to the top of grid point (i, j) is blocked (or non-existent) iff $\mathsf{Top}[i, j] = 1$, and the path to the bottom of grid point (i, j) is blocked (or non-existent) iff $\mathsf{Bottom}[i, j] = 1$. As an example, Fig. 1 shows a 3×4 maze and the corresponding blocking matrices. The grid points are numbered $(1, 1)$ to $(3, 4)$.

A maze



Blocked paths indicated by ▨

Blocking Matrices

Top

1	1	1	1
0	1	0	1
0	0	1	0

Bottom

0	1	0	1
0	0	1	0
1	1	1	1

Left

1	0	0	1
1	1	0	1
1	1	0	0

Right

0	0	1	1
1	0	1	1
1	0	0	1

Figure 1: A maze and blocking matrices

Given (i) a 2−dimensional maze as described above, (ii) an entry grid-point and (iii) an exit grid-point that is distinct from the entry grid-point, we wish to determine if there exists a path without any blockages from the entry point to the exit point.

Inspired by our success in solving several puzzles using propositional logic in class and in the tutorials, we now want to construct a propositional logic formula ϕ from the three inputs labeled (i), (ii) and (iii) above such that:

- ϕ is satisfiable if and only if there exists a path without any blockages from the entry point to the exit point.
- The size of ϕ (no. of propositions and operators) is polynomial in the size of the maze grid.
- ϕ is of the form $\phi_{entry} \wedge \phi_{common} \wedge \phi_{exit}$, where

- ϕ_{common} depends only on the 2-dimensional maze and is independent of the entry and exit points.
- ϕ_{entry} depends only on the entry-point and on blocking matrix entries corresponding to the entry point. Specifically, ϕ_{entry} is independent of the exit point.
- ϕ_{exit} depends only on the exit-point and on blocking matrix entries corresponding to the exit point. Specifically, ϕ_{exit} is independent of the entry point.

Thus, if only the entry point is changed while keeping the maze and exit point unchanged, ϕ for the resulting problem should be obtainable simply by changing ϕ_{entry}. Similarly, if only the exit point is changed, ϕ should be obtainable simply by changing ϕ_{exit}.

(a) Give a set of propositions that can be used to solve the above problem for the maze depicted in Fig 1. You must indicate what the truth values assigned to your propositions denote when interpretted in the context of the original maze problem.

(b) Give ϕ, ϕ_{entry}, ϕ_{common} and ϕ_{exit} satisfying the conditions listed above. You may assume that the maze is as depicted in Fig 1, the entry point is $(3, 4)$ and the exit point is $(2, 1)$. Give brief justification for the construction of each of your formulae.

(c) Give ϕ_{entry} and ϕ_{exit} for the entry point $(3,1)$ and exit point $(1,4)$ as well. Remember that your formulae should be such that when these are conjoined with ϕ_{common} given by you above, we should get a formula ϕ that is satisfiable iff there exists a path without blockage from $(3,1)$ to $(1,4)$ in the same maze depicted in Fig. 1.

(d) If we are to apply the strategy used by you for the maze depicted in Fig 1, what will be the sizes of ϕ_{entry}, ϕ_{common} and ϕ_{exit} for a general $2-$dimensional $n \times m$ maze? The size of a formula can be considered as the number of propositions and operators appearing in the formula. You can use the big-Oh notation (like $O(n^2)$ or $O(\log n)$, etc.) to describe the sizes of formulae in this question.

Do NOT WRITE BEYOND THIS LINE.