Balanced Group Labeled Graphs

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Outline

Oriented Group Labeled Graphs

- Oriented graphs
- Edges labeled by elements of a group
- Label of a path in underlying undirected graph
- Add labels of edges in sequence
- Labels of oppositely oriented edges are subtracted
- Cycle has (non)-zero label independent of starting vertex

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Undirected Group Labeled Graphs

- Undirected graphs with loops and multiple edges
- Edges / vertices labeled by elements of an abelian group
- Labels are also called weights
- Weight of a subgraph
- Sum of weights of vertices and edges in the subgraph

Consider only undirected group labeled graphs

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Signed and Marked Graphs

- Signed graphs
  - Undirected graphs with edges labeled '$+$' or '$-$'
- Marked graphs
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- Special cases of Z_2-labeled graphs
- Well-studied in the literature
- Extend some results to general group labeled graphs

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Labellings with Specified Subgraphs of Weight Zero

- $\mathcal{F}$ is a family of graphs
- $\mathcal{F}$–balanced labellings of a graph G
- Every subgraph of G in $\mathcal{F}$ has weight zero
- Labellings form a group
- What is its order?

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Balanced Labellings

- $\mathcal{F}$ is the set of all cycles
- Labellings in which every cycle has weight zero — Balanced labellings
- Balanced signed graphs
- Consistent marked graphs
- Characterizations of such Z_2–labellings known
- Extend these to labellings by arbitrary abelian groups
- Count the number of balanced labellings

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Balanced Signed Graphs

Theorem (Harary, 1954)

A signed graph is balanced iff the vertex set can be partitioned into two parts such that an edge has a '$-$' sign if and only if it has an endvertex in each part.

Consistent Marked Graphs

Theorem (Hoede, 1992)

A marked graph is consistent iff

1. *Every fundamental cycle with respect to any fixed spanning tree T is balanced.*
2. *Any path in T that is the intersection of two fundamental cycles has endvertices with the same signs.*

Earlier characterizations, more complicated, given by Rao and Acharya.

Alternative Characterizations

Theorem (Roberts and Xu, 2003)

A marked graph is consistent iff

1. *Every cycle in some basis for the cycle space is balanced.*
2. *Every 3-connected pair of vertices has the same sign.*

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3-Connected Pairs of Vertices

Theorem (Roberts and Xu,2003)

The following statements are equivalent for any marked graph

1. *Every 3-connected pair of vertices have the same sign.*
2. *Every 3-edge-connected pair of vertices have the same sign.*
3. *For any spanning tree T, the endvertices of any path in T that is the intersection of two fundamental cycles, have the same sign.*

Balanced Group Labeled Graphs

Theorem

Let $w : V(G) \cup E(G) \rightarrow \Gamma$ be a labeling of a graph G by an arbitrary abelian group Γ. Then w is a balanced labeling iff

1. *Every cycle in some basis has weight zero.*
2. *For every 3-connected pair of vertices u, v and any path P between u and v, $2w(P) = w(u) + w(v)$.*

- Other characterizations extend similarly
- Replace the condition "have the same sign" by "$2w(P) = w(u) + w(v)$"
- Holds if labels are assigned to vertices and edges
- Linear-time algorithm to test balance

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Counting Number of Balanced Labellings

- Define a relation $\sim$ on $V(G)$
- $u \sim v$ iff $u = v$ or there exist three edge-disjoint paths between u and v in G
- $\sim$ is an equivalence relation on $V(G)$
- $\sigma(G)$ is the number of equivalence classes of $\sim$
- $c(G)$ is the number of connected components of G

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Theorem

The number of distinct balanced labellings of a graph G by a finite abelian group Γ is

$$|\Gamma|^{|G|+\sigma(G)-c(G)}$$

- Depends only on the order and not the structure of Γ
- If Γ is Z_2 this follows from the characterization
- Sufficient to prove it for cyclic groups Z_k and 2-edge-connected graphs

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Lemma

Let w be a labeling of a 3-edge-connected graph G. Then w is balanced iff for any two vertices u, v and path P between u and v, $2w(P) = w(u) + w(v)$.

- Sufficient to prove it for edges
- To prove for edges, sufficient to prove for 3 edge-disjoint paths with same endvertices
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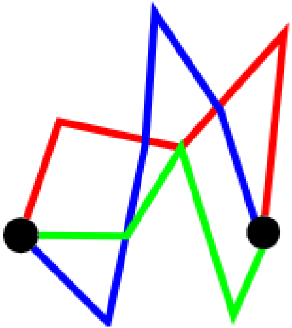

3-Edge-Connected Graphs

- If k is odd, in any balanced labeling by Z_k, labels of vertices uniquely determine the labels of edges
- Labels of vertices may be arbitrary
- Number of labellings is $k^{|G|}$

3-Edge-Connected Graphs

- If k is even, all vertex labels must have same parity
- Two possible choices of $w(uv)$ such that $2w(uv) + w(u) + w(v) = 0$
- Choices cannot be made arbitrarily
- There is a partition of the vertex set into two parts such that $w(uv) = -\frac{w(u)+w(v)+k}{2}$ iff u, v are in different parts
- Number of labellings is $2(k/2)^{|G|} \times 2^{|G|-1} = k^{|G|}$

2-Edge Connected Graphs

- Simple inductive argument
- Consider 2-edge cut X such that size of smaller component of $G - X$ is minimum
- Apply Lemma for 3-edge connected graphs to this component if it is non-trivial
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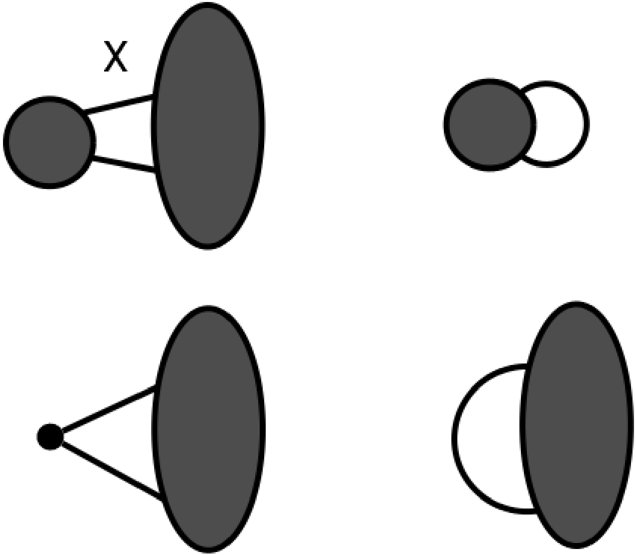

Balanced Labellings with Some Edges Labeled Zero

Lemma

Let G be a 3-edge-connected graph and F a subset of edges of G. Let $c(F)$ denote the number of connected components of the spanning subgraph of G with edge set F, and let $b(F)$ be the number of these components that are bipartite. The number of balanced labellings of G by Z_k, with edges in F having label 0 is

1. $k^{b(F)}$ *if k is odd or k is even and $c(F) = b(F)$.*
2. $k^{b(F)}2^{c(F)-b(F)-1}$ *if k is even and $c(F) > b(F)$.*

- Argument extends to 2-edge-connected graphs
- Efficient algorithm for counting such labellings

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Consistently Markable Graphs

- Which graphs have a non-trivial balanced labeling by Z_2 with all edge labels zero? (Beineke and Harary, 1978)
- Roberts (1995) characterized all such 2-connected graphs with longest cycle of length at most 5.
- Constructive characterization of all such graphs
- Follows from the inductive characterization of balanced labellings

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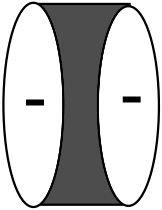
Characterization of Markable Graphs

Theorem

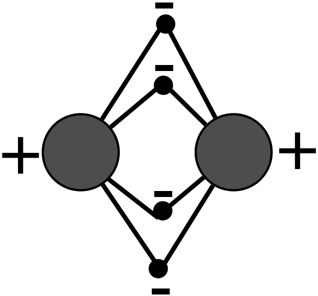
A 2-edge-connected graph G is markable if and only if it satisfies one of the following properties.

(a) G is bipartite.



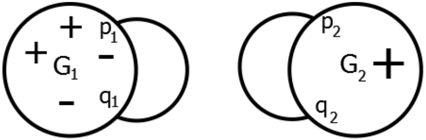
Characterization of Markable Graphs

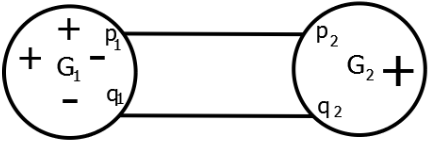
(b) There is a 3-edge-connected graph G' and a non-empty proper subset $\emptyset \subset A \subset V(G')$, such that G is obtained by subdividing exactly once every edge in the cut $(A, \overline{A})$.



Characterization of Markable Graphs

(c) G is obtained from the disjoint union of a 2-edge-connected markable graph G_1 and an arbitrary 2-edge-connected graph G_2, by replacing edges $p_i q_i \in E(G_i)$ by edges $p_1 p_2$ and $q_1 q_2$.





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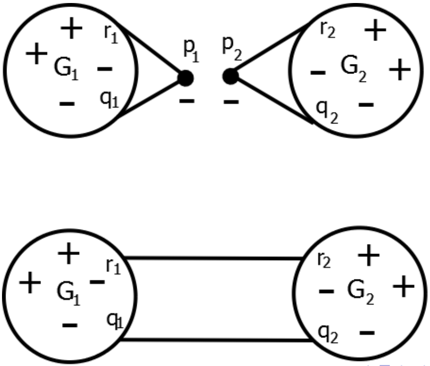
(d) G is obtained from the disjoint union of two 2-edge-connected markable graphs G_1 and G_2 by deleting vertices $p_i \in V(G_i)$ of degree 2 and adding edges $q_1 q_2$, $r_1 r_2$, where q_i, r_i are the neighbors of p_i in G_i.



Conclusions

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- Count the number of balanced labellings with specified elements labeled 0.
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- Extend to balanced labellings of other subgraphs (perhaps disjoint cycles, r-regular graphs)
- Nowhere-zero balanced labellings (similar to nowhere-zero flows)
- A deletion-contraction recurrence for the number of nowhere-zero balanced labellings
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