Triangle-factors in powers of graphs

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Abstract

The kth power of a graph G, denoted G^k, is the graph with the same vertex set as G, and two vertices are adjacent in G^k iff the distance between them in G is at most k. It is well-known that G^3 is Hamiltonian for any connected graph G. Also, G^2 is Hamiltonian for any 2-connected graph and for any connected claw-free graph G.

In this paper we investigate the existence of other 2-factors in powers of graphs. In particular, we consider triangle-factors, that is, 2-factors in which every cycle is of length 3. We assume implicitly that the order of the graph is divisible by 3, whenever it is necessary. It is easy to show that G^4 contains a triangle-factor for any connected graph and G^3 contains a triangle-factor for any connected claw-free graph G. Our main result is that G^3 contains a triangle-factor for any 2-connected graph G. We also show that any 2-connected claw-free graph contains a P_3-factor which implies that the square of the graph contains a triangle-factor. These results do not hold in general for smaller powers of graphs but we believe it may be possible to show that the square of any 3-connected graph contains a triangle-factor.

1 Introduction

The study of Hamiltonian graphs is one of the fundamental topics in graph theory and many classical results, such as Dirac's theorem, give sufficient conditions for a graph to be Hamiltonian [2]. One such result is that the cube of any connected graph is Hamiltonian [7, 10]. An important result on Hamilton cycles is Fleischner's theorem [3] that the square of any 2-connected graph contains a Hamilton cycle.

In this paper, we investigate the existence of other 2-factors in powers of graphs. In particular, we consider triangle-factors, that is, 2-factors in which every cycle is of length 3. Analogous to Dirac's minimum degree condition for Hamilton cycles, Corrádi and Hajnal [1] showed that every graph of order $3k$ and minimum degree $\geq 2k$ contains a triangle-factor. Recently, Wang [11] showed that the same conclusion is true if the sum of degrees of non-adjacent vertices is at least $4k-1$, giving an analogue of Ore's theorem for Hamilton cycles.

It is not difficult to show that the fourth power of a connected graph of order $3k$ contains a triangle-factor. Our main result is that the cube of a 2-connected graph of order $3k$ contains a triangle-factor, giving an analogue of Fleischner's theorem for Hamilton cycles.

We also consider claw-free graphs. It is known that the square of any connected claw-free graph is Hamiltonian [8]. It is easy to show that the cube of any connected claw-free graph of order $3k$ contains a triangle-factor. We show that any 2-connected claw-free graph of order $3k$ contains a P_3-factor and hence its square contains a triangle-factor. This result has been recently obtained by Kaneko et. al. [6], however we give different and shorter proofs of this and some other results in that paper. This also generalizes a result of Jünger et. al. [5] on partitioning the edges of a 2-edge-connected graph into connected parts and implies the result of Gernert [4] on the domination number of a 2-connected claw-free graph.

These results do not hold in general for smaller powers of graphs, but we believe it may be possible to show that the square of any 3-connected graph of order $3k$ contains a triangle-factor.

2 Notation

In this section we briefly describe our notation and terminology. Most of it is standard and may be found in standard books such as [2]. The vertex set of a graph G is denoted by $V(G)$ and the order of G, denoted $|G|$, is the number of vertices in G. We do not always distinguish between a graph and its vertex set when the meaning is clear from the context. The subset of vertices in G which are adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. If $S \subset V(G)$, $G-S$ is the subgraph of G obtained by deleting all vertices in S and edges incident to them. The subgraph of G induced by S is $G-V(G)\setminus S$. A *neighbour* of S is a vertex not in S that is adjacent to some vertex in S. When S contains a single vertex, we write v instead of $\{v\}$.

The *kth* power of a graph G, denoted G^k, is the graph with the same vertex set as G, and two vertices are adjacent in G^k iff the distance between them in G is at most k. A *factor* of a graph G is a spanning subgraph of G. If H is a connected graph, an H-factor of G is a factor of G in which every connected component is isomorphic to H. If $H = K_3$, we call an H-factor a triangle-factor and denote it by Δ-factor. A *claw* in a graph is an induced subgraph isomorphic to $K_{1,3}$ and a graph which does not contain a claw is said to be claw-free.

3 Connected graphs

In this section, we prove the simple results for existence of triangle-factors in powers of connected and connected claw-free graphs. We also give examples to show that smaller powers do not suffice for connected graphs in general.

Lemma 1 *The fourth power of any connected graph of order* $3k$ *contains a* Δ*-factor.*

Proof: The proof is by induction on the number of vertices in the graph G. If $|G| = 3$, then G^4 is a triangle. We will show that if $|G| > 3$ we can delete three vertices from G, which form a triangle in G^4, such that the resulting graph is connected. Let T be any spanning tree in G and let $P = v_1, \ldots, v_k$ be a longest path in T. Any vertex adjacent to v_2 in T except v_3 must be a leaf. If there are at least two such leaf vertices adjacent to v_2, we can delete either three leaves or two leaves along with v_2. If v_1 is the only leaf vertex adjacent to v_2 and v_3 has a neighbour v in T that is not in P, delete v_1, v_2 and v if v is a leaf else delete v_1, v_2 and a leaf vertex adjacent to v. If there is no such vertex v, delete v_1, v_2, v_3. $\square$

Lemma 2 *The cube of any connected claw-free graph of order $3k$ contains a Δ-factor.*

Proof: The proof is similar to that for fourth power of connected graphs. Again, let T be a spanning tree in the graph G and $P = v_1, \ldots, v_k$ be a longest path in T. If there are two or more leaf vertices adjacent to v_2, delete either three leaves or two leaves and v_2 to get a smaller connected claw-free graph. Suppose v_1 is the only leaf vertex adjacent to v_2 and there is a vertex v not in P that is adjacent to v_3 in T. Either v is adjacent to v_2 or v_4 in G or v_2 is adjacent to v_4 in G, otherwise v_2, v_3, v_4, v induce a claw in G. If v is a leaf vertex in T, delete v_1, v_2, v to get a smaller connected claw-free graph. Suppose v is not a leaf but is adjacent to a leaf u in T. If it is adjacent to more than one leaf vertex, we can use the earlier argument by labeling u as v_1 and v as v_2. If v is adjacent to v_2, delete v_1, v_2, u . If there is another vertex w not in P that is adjacent to v_3 in T, we can apply the above argument, perhaps after relabeling some of the vertices. If v is the only vertex not in P that is adjacent to v_3 in T, and it is also adjacent to v_4 in G, delete v_1, v_2, v_3 else if v_2 is adjacent to v_4 in G delete u, v, v_3. Finally, if there is no such vertex v, delete v_1, v_2, v_3. In all the cases, the deleted vertices form a triangle in G^3 and the remaining graph is connected and claw-free. $\square$

These results do not hold for smaller powers of graphs. Let G be the graph of order 9 obtained by subdividing every edge of $K_{1,4}$. There are 4 vertices of degree 1 in G at distance 4 from each other. Clearly, G^3 cannot contain 3 vertex-disjoint triangles. Similarly, let G be the line-graph of the graph obtained by subdividing every edge of $K_{1,3}$. The graph G is claw-free and contains 3 vertices of degree 1 at distance 3 from each other. Clearly, G^2 cannot contain two vertex-disjoint triangles. In the next sections, we show that assuming G to be 2-connected gives an improvement in the results.

4 Cube of 2-connected graphs

In this section, we prove that the cube of any 2-connected graph G of order divisible by 3 contains a Δ-factor. The proof is similar to Říha's [9] proof of Fleischner's theorem, an exposition of which may be found in [2]. The main idea is to start with a suitably chosen cycle C in G and consider the components of the graph $G-C$. We will show that for every such component D, we can find vertex-disjoint triangles in D^3 such that any vertex not in a triangle is a neighbour of C. The remaining vertices in each component can be combined with vertices in C to get the remaining triangles. We now present the proof formally in the form of three lemmas.

Lemma 3 *Let x be any vertex in a 2-connected graph G. There exists a cycle C in G containing x such that at least one of the following holds:*
(i) $G - C$ contains an isolated vertex y.
(ii) There exist two consecutive vertices y_1, y_2 in C, different from x, such that $N_G(y_1)$, $N_G(y_2) \subseteq V(C)$.

Proof: If G is Hamiltonian, we may choose C to be a Hamilton cycle, and y_1, y_2 to be any two consecutive vertices in C different from x. Suppose G is not Hamiltonian. Choose a cycle C containing x and a component D in $G - C$ such that $|D|$ is as small as possible amongst all possible choices for C and D. If $|D|$ is one, the vertex y in D is isolated in $G - C$ and we get the required cycle C. Suppose $|D| > 1$. Since G is 2-connected, C contains at least two neighbours of D. We can therefore choose two neighbours x_1, x_2 of D in C such that one of the paths, say P, between x_1 and x_2 in C does not contain x or any other neighbour of D as an internal vertex. Let P' be a path between x_1 and x_2 such that all internal vertices of P' are in D. Let C' be the cycle obtained by replacing P by P' in C. P' must contain all vertices in D otherwise $G - C'$ contains a component that is a proper subgraph of D. Since $|D| > 1$, we can choose y_1, y_2 to be any two consecutive internal vertices in P' and C' is the required cycle. $\square$

We say a tree T is a *caterpillar* if there exists a path $v_1, v_2, \ldots, v_k$, $k \geq 1$, in T such that every vertex not in the path is a leaf vertex. Every such vertex must be adjacent to exactly one vertex in the path and we denote by V_i the set of leaf vertices of T which are adjacent to v_i, $1 \leq i \leq k$. We say a caterpillar is *bad* if k is odd, $|V_i| \equiv 1 \bmod 3$ for all odd i and V_i is empty for all even i, $1 \leq i \leq k$.

Lemma 4 *Let T be a caterpillar of order n. Then the following hold:*
(i) If $n \equiv 0 \bmod 3$, T^3 contains a Δ-factor.
(ii) If $n \equiv 1 \bmod 3$, $T^3 - v_1$ contains a Δ-factor.
(iii) If $n \equiv 2 \bmod 3$, $T^3 - \{v_1, v_k\}$ contains a Δ-factor unless T is bad.

Proof: The proof is by induction on n. The result is trivial for $n = 1$. If $n = 2$, either $k = 2$ in which case it is trivial or $k = 1$ and $|V_1| = 1$, in which case T is bad and $T^3 - \{v_1, v_k\}$ contains a single vertex and hence no Δ-factor. In either case, statement (iii) is satisfied. If $n = 3$, T^3 is itself a triangle. Suppose $n > 3$. We will show that we can delete 3 vertices forming a triangle in T^3 such that the remaining forest is a caterpillar. We consider different cases based on n mod 3.

Case 1 $n \equiv 0 \bmod 3$.

If $|V_k| \geq 3$, delete any 3 vertices from V_k to get a caterpillar T'. By induction, T'^3 contains a Δ-factor and the three deleted vertices form a triangle in T^3. This gives a Δ-factor in T^3.

If $|V_k| = 2$, delete the two vertices in V_k and v_k and apply the same argument.

If $|V_k| = 1$, then $k \geq 2$. Delete v_k and the only vertex in V_k and let T' be the resulting caterpillar. Since $|T'| \equiv 1 \bmod 3$, by induction and relabeling the vertices, we get a Δ-factor in $T'^3 - v_{k-1}$. Since v_k, v_{k-1} and the vertex in V_k form a triangle in T^3, we get the required Δ-factor in T^3.

Finally, if $|V_k| = 0$, delete v_k and a vertex in V_{k-1} if it is non-empty or v_{k-1} otherwise and apply the same argument as in the case when $|V_k| = 1$.

Case 2 $n \equiv 1 \bmod 3$.

Again, if $|V_k| \geq 2$, delete any three vertices in V_k if $|V_k| \geq 3$ or two vertices in V_k along with v_k otherwise and apply induction.

If $|V_k| = 1$, delete the only vertex in V_k, v_k and a vertex in V_{k-1} if it is non-empty or v_{k-1} otherwise and apply induction.

Finally, if $|V_k| = 0$ and $|V_{k-1}| \geq 2$, delete v_k and two vertices in V_{k-1} and apply induction. If $|V_{k-1}| = 1$, delete v_k, v_{k-1} and the vertex in V_{k-1} and apply induction. If V_{k-1} is also empty, delete v_k, v_{k-1} and a vertex in V_{k-2} if it is non-empty or v_{k-2} otherwise and apply induction.

Case 3 $n \equiv 2 \bmod 3$.

If $|V_k| \geq 3$, delete 3 vertices from V_k to get a caterpillar T'. By induction, either $T'^3 - \{v_1, v_k\}$ has a Δ-factor or else T' is bad. If T' is bad then so is T. Thus we get a Δ-factor in $T^3 - \{v_1, v_k\}$ unless T is bad.

If $|V_k| = 2$ and $|V_{k-1}| \geq 1$, delete the two vertices in V_k and any one vertex in V_{k-1}. The resulting caterpillar T' is not bad and we get a Δ-factor in $T'^3 - \{v_1, v_k\}$. Adding the triangle formed by the deleted vertices gives a Δ-factor in $T^3 - \{v_1, v_k\}$. If V_{k-1} is empty, delete v_k, v_{k-1} and the two vertices in V_k to get a caterpillar T'. Since $|T'| \equiv 1 \bmod 3$, $T'^3 - v_1$ contains a Δ-factor. Adding the triangle formed by v_{k-1} and the two vertices in V_k gives a Δ-factor in $T^3 - \{v_1, v_k\}$.

If $|V_k| = 1$ and $|V_{k-1}| \geq 2$, delete the vertex in V_k and two vertices in V_{k-1} and apply induction. If $|V_{k-1}| = 1$, delete the vertices v_k, v_{k-1} and the two vertices in $V_k \cup V_{k-1}$ and apply the same argument as above. If V_{k-1} is empty, delete the vertex in V_k and the vertices v_k and v_{k-1}. If the resulting caterpillar T' is not bad, there is a Δ-factor in $T'^3 - \{v_{k-2}, v_1\}$ which together with the triangle formed by v_{k-2}, v_{k-1} and the vertex in V_k gives a Δ-factor in $T^3 - \{v_1, v_k\}$. On the other hand, if T' is bad then so is T.

Finally, if V_k is empty, delete the vertex v_k and apply induction to the resulting caterpillar.

This completes the proof of the lemma. □

Lemma 5 *Let G be any 2-connected graph and v a specified vertex in G. Then there exists a subset $S \subseteq N_G(v)$ such that $(G - v)^3 - S$ contains a Δ-factor.*

Proof: The proof is again by induction on the order of G. If $|G| = 3$, then G is a triangle and we choose S to be the two neighbours of v. Suppose $|G| > 3$ and let $C = v, v_1, \ldots, v_k$ be a cycle in G satisfying the properties stated in lemma 3. If C is a Hamilton cycle, we can choose $S \subseteq \{v_1, v_k\}$, such that $|(G - v) - S| \equiv 0 \bmod 3$. Since $(G - v) - S$ contains a Hamilton path, $(G - v)^3 - S$ contains a Δ-factor.

Suppose G is not Hamiltonian and let D be any non-trivial component in $G - C$. Let G' be the graph obtained from G by contracting $G - D$ to a vertex v'. The graph G' is 2-connected and has order less than $|G|$. Therefore, by induction, there exists a subset $S_D \subseteq N_{G'}(v')$, such that $(G' - v')^3 - S_D$ contains a Δ-factor. Since $(G' - v') = D$ is a subgraph of G not containing v, all these triangles are contained in $(G - v)^3$ and $S_D \subseteq V(D)$. Let S' be the union of all the sets S_D for all non-trivial components D and all the isolated vertices in $G - C$. Then $(G - v)^3 - (S' \cup \{v_1, \ldots, v_k\})$ contains a Δ-factor formed by the union of the Δ-factors in the graphs $D^3 - S_D$ for all non-trivial components D in $G - C$.

Let G'' be the subgraph of G induced by $V(C) \cup S'$ and let S'' be the subset of vertices in S' that are adjacent to v. The graph $(G'' - v) - S''$

contains a path $v_1, \ldots, v_k$ such that every vertex not in the path is adjacent to some vertex in the path and hence contains a caterpillar T. The cycle C is chosen to satisfy the properties stated in lemma 3 and hence either T contains two vertices v_i, v_{i+1} such that V_i and V_{i+1} are empty or $G - C$ contains at least one isolated vertex.

In the first case, T cannot be bad and by lemma 4 we can choose a subset $S''' \subseteq \{v_1, v_k\}$ such that $T^3 - S'''$ contains a Δ-factor. Combining this with the Δ-factor in $(G-v)^3 - (S' \cup \{v_1, \ldots, v_k\})$, we get a Δ-factor in $(G-v)^3 - (S'' \cup S''')$. Since $S'' \cup S''' \subseteq N_G(v)$, we get the required subset $S = S'' \cup S'''$.

If $G - C$ contains an isolated vertex u, it is possible that T is bad. However, u is adjacent to at least two vertices in C since G is 2-connected. If $u \in S''$, we add u to T and an edge from u to a neighbour of u in C different from v. If $u \notin S''$ and is adjacent to v_i in T, we make it adjacent to $v_j, j \neq i$ instead. In either case we get a new caterpillar which cannot be bad and the same argument can be used to get the required set S. □

We can now prove the main theorem.

Theorem 1 *The cube of any 2-connected graph of order $3k$ contains a Δ-factor.*

Proof: The proof follows from lemma 5, since the cube of a subgraph induced by a vertex and its neighbours contains a Δ-factor if its order is divisible by 3.

5 Claw-free graphs

In this section, we prove that every 2-connected claw-free graph of order divisible by 3 contains a P_3-factor, which implies that its square contains a Δ-factor. The proof is similar to that for the cube of 2-connected graphs and is different from that given in [6]. This also generalizes the result of Jünger et. al. [5], who showed that the edges of a 2-edge-connected graph of size divisible by 3 can be partitioned into connected parts of size 3. Also, this implies the result of Gertnert [4] that the domination number of any 2-connected claw-free graph of order n is at most $\lceil n/3 \rceil$.

An *end-block* of a graph G is block of G containing at most one cut-vertex of G. Let $eb(G)$ denote the number of end-blocks of G. We also give a simple proof of the result in [6] that any connected claw-free graph G with $eb(G) \geq 2$ contains $\lfloor (|G| - eb(G) + 2)/3 \rfloor$ vertex-disjoint P_3.

We first introduce some terminology to aid in the proof. The vertex of degree 3 in any induced claw is said to be the *center* of the claw. We say a graph G is *nearly-claw-free* if there is a vertex v in G such that every induced claw in G has v as its center. The vertex v is said to be the *base* of the nearly-claw-free graph.

Lemma 6 *Let G be a 2-connected nearly-claw-free graph with base v. There exists a subset $S \subseteq N_G(v)$ such that $(G - v) - S$ contains a P_3-factor.*

Proof: The proof is again by induction on the order of G. If every vertex is adjacent to v, we choose $S = N_G(v)$ and the result is trivial. Let u be a vertex not adjacent to v. Since G is 2-connected, there is a cycle including both u and v. Choose a vertex u not adjacent to v, and a cycle $C = v(= v_0), v_1, v_2, \ldots, v_k$ passing through both u and v such that the length of C is as small as possible.

Note that C cannot be a triangle since u is not adjacent to v, and v cannot be adjacent to any vertex other than v_1 and v_k in C. Also, if the length of C is at least 5, then v_{i-1} cannot be adjacent to v_{i+1}, $1 \leq i < k$, otherwise we can get a shorter cycle containing v and a vertex not adjacent to v.

If C is a Hamilton cycle, we can choose $S \subseteq \{v_1, v_k\}$ such that $(G-v)-S$ contains a P_3-factor. Let D be any non-trivial component in $G-C$. As before, contract $G - D$ to a vertex v' to get the graph G', which is 2-connected and also nearly-claw-free with base v'. By induction, there exists a subset $S_D \subseteq N_{G'}(v')$ such that $(G'-v')-S_D$ contains a P_3-factor. Since $(G'-v') = D$ is a subgraph of G, this P_3-factor is contained in $G - v$ also and $S_D \subseteq V(D)$.

Let S' be the union of all the sets S_D and the isolated vertices in $G - C$. Note that $(G-v)-(S' \cup \{v_1, \ldots .v_k\})$ contains a P_3-factor obtained by taking the union of all the P_3-factors in $D - S_D$, for all non-trivial components D is $G - C$. Also, every vertex in S' is a neighbour of C. Let $S_i, 0 \leq i \leq k$ be the subset of vertices in S' adjacent to v_i but not adjacent to any v_j for $j < i$. The subgraph of G induced by each S_i, for $1 \leq i \leq k$, must be complete, otherwise v_i, v_{i-1} and two non-adjacent vertices in S_i form a claw in G with center v_i. Also, if v_{i-1} is not adjacent to v_{i+1}, every vertex in S_i must be adjacent to v_{i+1}. Finally, note that S_k must be empty since any vertex not in C and adjacent to v_k must also be adjacent to either $v_0 = v$ or v_{k-1}.

If $|C| > 4$, we can find a path from v_1 to v_k, which includes all the vertices in $S_i \cup \{v_i\}$, for $1 \leq i \leq k$. If P_i is any Hamilton path in the complete graph induced by S_i then $v_1, P_1, v_2, P_2, \ldots, v_{k-1}, P_{k-1}, v_k$ is the required path. Thus we can choose a subset $S'' \subseteq \{v_1, v_k\}$ such that $(G - v) - (S_0 \cup S'')$ has a P_3-factor. Since $S_0 \cup S'' \subseteq N_G(v)$, we get the required subset $S = S_0 \cup S''$.

If $|C| = 4$ and v_1 is not adjacent to v_3, the same argument as above applies. If v_1 is adjacent to v_3, consider the path v_3, v_1, P_1, v_2, P_2, which includes all vertices in $S_1 \cup S_2$. Again, the same argument can be used to find a P_3-factor by deleting a subset of $\{v_1, v_3\}$ along with S_0. □

Theorem 2 *Any 2-connected claw-free graph of order $3k$ contains a P_3-factor.*

Proof: This follows from lemma 6 since, in any claw-free graph, the subgraph induced by a vertex and its neighbours contains a Hamilton path and hence a P_3-factor if its order is divisible by 3.

Theorem 3 *Let G be a connected claw-free graph with $eb(G) \geq 2$. Then G contains $\lfloor (|G| - eb(G) + 2)/3 \rfloor$ disjoint paths of order 3.*

Proof: Suppose, there is an end-block B of G with $|B| \geq 4$ and let v be the cut-vertex of G contained in B. Since B is 2-connected and claw-free, by lemma 6, there is a subset $S \subseteq N_B(v)$ such that $(B - v) - S$ has a P_3-factor. Since $N_B(v)$ induces a complete subgraph of G, we may assume $|S| \leq 2$. The result now follows by induction since $eb(G - V(B) \setminus (S \cup \{v\})) \leq eb(G)$.

If there is no such block B, every end-block of G is either K_2 or K_3. Let G' be the graph obtained by adding a new vertex v' to G and joining it to exactly one vertex, which is not a cut-vertex of G, in each end-block of G. The graph G' is 2-connected and also nearly-claw-free with base v'. By lemma 6, there is a subset $S' \subseteq N_{G'}(v')$ such that $(G' - v') - S'$ has a P_3-factor. The result now follows since $|S'| \leq eb(G)$. □

6 Conclusion

It is not difficult to construct 2-connected graphs of order $3k$ whose square does not contain a Δ-factor. However, we do not know of any examples of 3-connected graphs of order $3k$ whose square does not contain a Δ-factor. We believe it may be possible to show that the square of every 3-connected graph contains a Δ-factor. More generally, it would be interesting to study the relationship between connectivity and the smallest power of a graph containing a given subgraph.

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