On Synthesizing Computable Skolem functions for FO logic

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Skolem functions

Given a FOL formula $\varphi(X, Y)$ over (inputs) X and (outputs) Y, $F(\cdot)$ is a Skolem function iff

$$\forall X \left(\exists Y \varphi(X, Y) \Leftrightarrow \varphi(X, F(X))\right)$$

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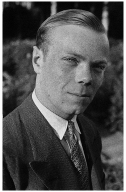
- Classical concept arising from quantifier elimination in FOL.
- Known to always exist! But,
  1. Is the function computable?
  2. Can we effectively compute/synthesize such a function?

A storied history



Skolem functions play an important role in first order logic

- Getting rid of existential quantifiers
- Seminal work by Thoralf Skolem 1920s and Jacques Herbrand 1930s.
- Skolemization and “Skolem-Normal form”
- Focus on existence of form, NOT computability.

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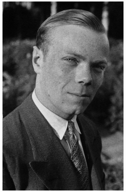

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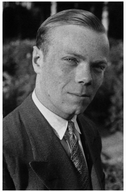

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- Existence and construction of Boolean unifiers
- Boole’1847, Lowenheim’1908.


Applications

Why should we be interested in synthesizability of Skolem functions?

- Heart of Automated Program Synthesis and repair.

$g(x_1, x_2) \geq x_1$ and
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$(g(x_1, x_2) == x_1$ or
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Golia et al, IJCAI'21

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Prior work

- Propositional setting: Akshay et al.'17,'18,'19,'20,'21, Rabe et al. '17,'18, Golia et al.'20,'21,etc., Fried et al'16, John et al.'15, Heule et al.'14, etc.
- Beyond Propositional setting:
  - Results on specific theories: Linear rational arithmetic Kuncak et al.'10, Bit vectors Spielman et al.,Priener et al.
  - Partial approach for Quantifier Elimination Jiang'09.

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Idea of going beyond terms not new: Skolem functions as set of conditional statements [Jiang'09]

Testing the limits

Can we always synthesize Skolem functions as Turing machines?

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- Unfortunately yes. Natural numbers over $\mathcal{V} = \{=, +, *, 0, 1\}$
- Follows from the classical Matiyasevich-Robinson-Davis-Putnam (MRDP) theorem!

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Note: We assume structures to be "computable": predicates/functions are effectively computable.

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Let $\mathfrak{M}$ be a computable $\mathcal{V}$-structure for vocabulary $\mathcal{V}$.

- SkSyn has a positive answer for $\mathfrak{M}$ iff the "elementary diagram" of $\mathfrak{M}$ is decidable.

So what is the elementary diagram of $\mathfrak{M}$?

A brief detour into Model theory

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Elementary diagram is said to be decidable if given any sentence φ in $\mathcal{V}(\mathfrak{M})$, we can algorithmically decide if $\varphi \in ED(\mathfrak{M})$.

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Elementary diagram is said to be decidable if given any sentence φ in $\mathcal{V}(\mathfrak{M})$, we can algorithmically decide if $\varphi \in ED(\mathfrak{M})$.

This is the necessary and sufficient condition for synthesis!

Consequences and more!

Theorem

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- If theory admits effective constraint solving, then can give upper bounds! (see paper)

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A framework to the study algorithmic computation of Skolem functions.

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Thank you!

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     3. if true, output ρ, quit loop, else goto next elt.

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4. For two distinct elements $d, e \in \mathfrak{M}$, evaluate $F(c, d, e)$.
   - if $F(c, d, e) = d$, then φ is valid.
   - else $F(c, d, e) = e$ and φ is not valid.

$\square$