CS 344
Artificial Intelligence
By Prof: Pushpak Bhattacharyya
Class on 07/Feb/2007
Proof of Deduction Theorem (D. T)

• Statement:
  If \( A_1, A_2, A_3, \ldots, A_n \vdash B \) then
  \( A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \to B \)

• Proof:
  **Case 1**: \( B \) is an axiom
  \( A_1, A_2, A_3, \ldots, A_{n-1}, B \)
  \( B \to (A_n \to B) \) \quad \ldots \quad \text{Axiom 1}
  \( A_n \to B \) \quad \text{by MP} \quad \ldots \quad \text{QED}
Proof of Deduction Theorem

Case 2: $B$ is one of $A_1, A_2, A_3, \ldots, A_n$

Part (a): $B$ is $A_n$

$A_1, A_2, A_3, \ldots, A_{n-1}, B$

$A_n \rightarrow A_n \ldots$ theorem

i.e., $A_n \rightarrow B \ldots$ QED

Part (b): $B$ is not $A_n$

$A_1, A_2, A_3, \ldots, A_{n-1}, A_i \ldots \ i = 1, \ldots, (n-1)$

$A_i \rightarrow (A_n \rightarrow A_i) \ldots$ by Axiom1

$A_n \rightarrow A_i$ i.e. $B \ldots$ QED
Proof of Deduction Theorem

Case 3: $B$ is the result of MP on $E_i$ and $E_j$, where $E_i$ and $E_j$ come before $B$ in the proof sequence and $E_j$ is $E_i \rightarrow B$

$A_1, A_2, A_3, \ldots, A_{n-1}, E_i, \ldots, E_j, \ldots, B$

Now if

$A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \rightarrow E_i$ is true

and

$A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \rightarrow E_j$ is true, i.e.,

$(A_n \rightarrow (E_i \rightarrow B))$ is true

then by Axiom-2 and MP we have

$A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \rightarrow B$
Proof of Deduction Theorem
(contd.)

Showing $A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \rightarrow E_i$ depends on
proving DT using a shorter segment of $A_1, A_2, A_3, \ldots, A_n \vdash B$

Similarly for $A_1, A_2, A_3, \ldots, A_{n-1} \vdash A_n \rightarrow E_j$

As this continues we come to a situation where $E_i (E_j)$ must be a
hypothesis or axiom

These cases have already been proved. QED
Fundamental Issues

- Soundness
- Completeness
- Consistency
- Decidability
- Complexity
Introducing Semantics into the Formal System

- **Valuation function** in Propositional Calculus (PC)
  - The *valuation function* is defined as:
    \[ V(E) \rightarrow \{ T, F \} \]
    where \( E \) is a well formed formula and \( T, F \) are symbols called `true’ and `false’ respectively.

- **Definition:**
  1. \( V(\mathcal{F}) = F \)
  2. \( V(P) = T \) or \( F \) where \( P \) is a proposition.
  3. \( V(P \rightarrow Q) \) is given by the table called `truth table’.
Introducing Semantics into the Formal System

<table>
<thead>
<tr>
<th>$V(P)$</th>
<th>$V(Q)$</th>
<th>$V(P \rightarrow Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>F</td>
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<td>T</td>
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</tbody>
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- **Definition:**
  - A tautology is a well formed formula whose valuation is always $T$ whatever be the valuation of its constituent propositions.
  - Such formulae are also called valid formulae. A formula which is true for some valuation is called satisfiable.
  - An unsatisfiable formula is also called invalid.

- **Soundness question:**
  - Is every theorem in the system a tautology?

- **Completeness question:**
  - Is every tautology a theorem of the system?