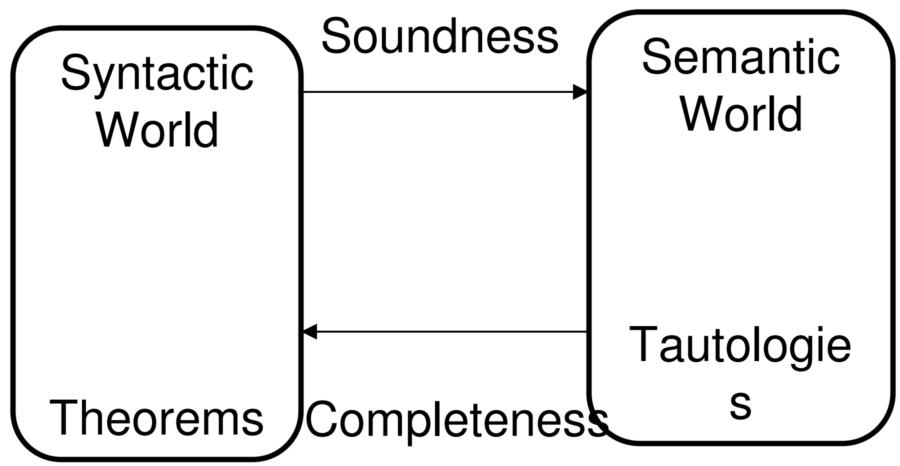
CS 344
Artificial Intelligence

By Prof: Pushpak Bhattacharya

Class on 12/Feb/2007

Soundness and Completeness

- Soundness: Correctness or trustworthiness of the system
- Completeness: Power of the system
- Consistency: The formal system should never derive *P* and ~*P* *i.e.* $\mathcal{F}$
- Soundness and consistency are related.
- Unsoundness implies inconsistency *i.e.* an unsound system can derive $\mathcal{F}$



Proof of Soundness of Propositional Calculus (PC)

- Theorems in PC are tautologies.
- <u>Sanity Check</u>: Are the three axioms tautologies?

  <u>Axiom 1 (A1)</u>: $A \rightarrow (B \rightarrow A)$

  Four models: *(T, T), (T, F), (F, T), (F, F)*

  A1 is true in all four models, thus *A1* is a tautology.

Proof of Soundness of Propositional Calculus (PC) (contd.)

Axiom 2 (A2): $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

8 models and *A2* is true in all models

Axiom 3 (A3): $((P \rightarrow \mathcal{F}) \rightarrow \mathcal{F}) \rightarrow P$

2 models and *A3* is true in both models

Exercise: Prove that MP meets the sanity check.

Proof of Soundness of Propositional Calculus (PC) (contd.)

- Theorem:

  If $A_1, A_2, A_3, \ldots, A_n$ |-- B then for every V in which $V(A_i) = true$ □ i, $V(B) = true$

- Proof:

  Case 1: B is an axiom

  In this case $V(B)$ is always true. Hence proved.

  Case 2: B is one of $A_1, A_2, A_3, \ldots, A_n$

  Since it is given that $V(A_i) = true$ □ i, for the given V, hence finally $V(B)$ is true.

Proof of Soundness of Propositional Calculus (PC) (contd.)

<u>Case 3</u>: *B* is neither an axiom nor a hypothesis.
B is the result of MP on E_i and E_j which come before B.

Assume $V(B) = false$; *B* came from E_i and $E_i \rightarrow B$ *(which is* E_j*)*

If $V(E_i) = true$ then $V(E_j) = false$ and if $V(E_i) = false$ then $V(E_j) = true$

This means one of E_i and E_j is not an axiom or hypothesis

This means that the false formula has to be the result of MP between E_m and E_n which come before E_i (or E_j)

Thus we go on making the value false for expressions which are closer and closer to the beginning of the proof. Eventually we have to assert that an axiom or hypothesis is false which leads to contradiction.

Relation between Soundness and Consistency

- An unsound system is consistent. Let us make PC unsound.
- For this, introduce $(A \square B)$ as an axiom.
- $(A \square B)$ can be made false by making $V(A)$ = false = $V(B)$
- Now the system will derive $\mathcal{F}$ as follows:

  $(\mathcal{F} \rightarrow \mathcal{F}) \rightarrow \mathcal{F}$ (new axiom with $\mathcal{F}$ for A and B)

  $((\mathcal{F} \rightarrow \mathcal{F}) \rightarrow \mathcal{F}) \rightarrow \mathcal{F})$ (axiom A3 with $\mathcal{F}$ for A)

  $\mathcal{F}$, MP