CS 344
Artificial Intelligence
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Soundness and Completeness

- **Soundness**: Correctness or trustworthiness of the system
- **Completeness**: Power of the system
- **Consistency**: The formal system should never derive $P$ and $\neg P$ i.e. $\mathcal{F}$
- Soundness and consistency are related.
- Unsoundness implies inconsistency *i.e.* an unsound system can derive $\mathcal{F}$

![Diagram showing Soundness and Completeness](attachment:image.png)
Proof of Soundness of Propositional Calculus (PC)

- Theorems in PC are tautologies.
- **Sanity Check**: Are the three axioms tautologies?
  
  **Axiom 1 (A1):** $A \rightarrow (B \rightarrow A)$

  Four models: $(T, T), (T, F), (F, T), (F, F)$

  $A1$ is true in all four models, thus $A1$ is a tautology.
Proof of Soundness of Propositional Calculus (PC) (contd.)

Axiom 2 (A2): \((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\)

8 models and A2 is true in all models

Axiom 3 (A3): \(((P \rightarrow \mathcal{F}) \rightarrow \mathcal{F}) \rightarrow P\)

2 models and A3 is true in both models

Exercise: Prove that MP meets the sanity check.
Theorem:
If $A_1, A_2, A_3, \ldots, A_n \vdash B$ then for every $V$ in which $V(A_i) = true \quad i, \quad V(B) = true$

Proof:

Case 1: $B$ is an axiom
In this case $V(B)$ is always true. Hence proved.

Case 2: $B$ is one of $A_1, A_2, A_3, \ldots, A_n$
Since it is given that $V(A_i) = true \quad i, \quad$ for the given $V$, hence finally $V(B)$ is true.
Proof of Soundness of Propositional Calculus (PC) (contd.)

Case 3: $B$ is neither an axiom nor a hypothesis. $B$ is the result of MP on $E_i$ and $E_j$ which come before $B$.

Assume $V(B) = false$; $B$ came from $E_i$ and $E_i \rightarrow B$ (which is $E_j$)

If $V(E_i) = true$ then $V(E_j) = false$ and if $V(E_i) = false$ then $V(E_j) = true$

This means one of $E_i$ and $E_j$ is not an axiom or hypothesis

This means that the false formula has to be the result of MP between $E_m$ and $E_n$ which come before $E_i$ (or $E_j$)

Thus we go on making the value false for expressions which are closer and closer to the beginning of the proof. Eventually we have to assert that an axiom or hypothesis is false which leads to contradiction.
Relation between Soundness and Consistency

- An unsound system is consistent. Let us make PC unsound.
- For this, introduce \((A \lor B)\) as an axiom.
- \((A \lor B)\) can be made false by making \(V(A) = false = V(B)\)
- Now the system will derive \(F\) as follows:
  \[
  (F \rightarrow F) \rightarrow F \quad \text{(new axiom with } F \text{ for } A \text{ and } B) \\
  ((F \rightarrow F) \rightarrow F) \rightarrow F \quad \text{(axiom A3 with } F \text{ for } A) \\
  F, \text{ MP}
  \]