CS 344
Artificial Intelligence

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Completeness of Propositional Calculus

- Statement:

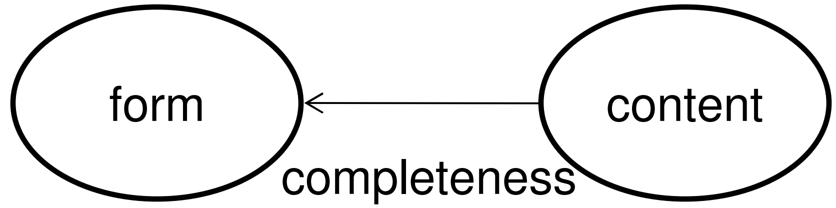
If *V(A) = true* for $\forall$ *V* for a well formed formula (wff) *A*,

then |-- *A, i.e.,*

there exists a proof for the theoremhood of A.

- Intuition through an example:
  - It is easy to show that

  $(p \rightarrow p \vee q)$ is a tautology.
  - Every row in the truth table has a ‘T’ at the end column.



p	***q***	***$p \rightarrow p \vee q$ (A)***
T	T	T
F	T	T
T	F	T
F	F	T

Completeness of Propositional Calculus (contd.)

- Treat every row of the truth table as follows:
  - If the value of a constituent proposition or *A* itself is false then replace it by negation of the entity, else leave it as such. Then set up derivations.
  - Row 1:

    $p, q \quad |\text{--} \quad p \rightarrow p \lor q$
  - Row 2:

    $\sim p, q \quad |\text{--} \quad p \rightarrow p \lor q$
  - Row 3:

    $p, \sim q \quad |\text{--} \quad p \rightarrow p \lor q$
  - Row 4:

    $\sim p, \sim q \quad |\text{--} \quad p \rightarrow p \lor q$

Completeness of Propositional Calculus (contd.)

- Proof of each row by DT
- Row 1:

  $p, q, p, p \rightarrow \mathcal{F} \quad |\text{--} \quad q$

  $p, q, p \quad |\text{--} \quad ((p \rightarrow \mathcal{F}) \rightarrow q)$
- Row 2:

  $\sim p, q, p, p \rightarrow \mathcal{F} \quad |\text{--} \quad q$
- Row 3:

  $p, \sim q, p, \sim p, \sim q \quad |\text{--} \quad \mathcal{F}$
- Row 4:

  $\sim p, \sim q, p, \sim q \quad |\text{--} \quad \mathcal{F}$

Completeness of Propositional Calculus (contd.)

We have shown:

1. $p, q \;|\text{--}\; p \rightarrow p \square q$ (1)
2. $\sim p, q \;|\text{--}\; p \rightarrow p \square q$ (2)
3. $p, \sim q \;|\text{--}\; p \rightarrow p \square q$ (3)
4. $\sim p, \sim q \;|\text{--}\; p \rightarrow p \square q$ (4)

Let $(p \rightarrow (p \square q))$ be denoted by A.

Completeness of Propositional Calculus (contd.)

- From (1) and (2),

$p \;\; |\text{--} \;\; (q \rightarrow A)$

and $p \;\; |\text{--} \;\; (\sim q \rightarrow A)$

$|\text{--} \;\; (q \rightarrow A) \rightarrow ((\sim q \rightarrow A) \rightarrow A)$

previously proved theorem

$|\text{--} \;\; ((\sim q \rightarrow A) \rightarrow A)$

$|\text{--} \;\; A$

□ $p \;\; |\text{--} \;\; A$ ….(5)

Completeness of Propositional Calculus (contd.)

- From (3) and (4)

  $\sim p \quad |\text{--} \quad A \quad ....(6)$

- From (5) and (6)

  $|\text{--} \quad A \quad ....(7)$

- Note the progressive dropping of propositions leading to (7)
- Also note for any formula each row is a derivation.

p	***q***	***p □ q (A)***
T	T	T
F	T	F
T	F	F
F	F	F