CS 344
Artificial Intelligence
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**Admissibility of A**

A* always terminates finding an optimal path to the goal if such a path exists.

**Intuition**

1) In the open list there always exists a node \( n' \) such that \( f(n) \leq f^*(S) \).

2) If A* does not terminate, the \( f \) value of the nodes expanded become unbounded.

1) and 2) are together inconsistent

Hence A* must terminate
Lemma
Any time before A* terminates there exists in the open list a node $n'$ such that $f(n') \leq f^*(S)$

For any node $n_i$ on optimal path,

$$f(n_i) = g(n_i) + h(n_i) \quad \text{Also } f^*(n_i) = f^*(S) \leq g(n_i) + h^*(n_i)$$

Let $n'$ be the first node in the optimal path that is in OL. Since all parents of $n'$ have gone to CL,

$$g(n') = g^*(n')$$

$$f(n') \leq f^*(S)$$
If A* does not terminate

Let $e$ be the least cost of all arcs in the search graph.

Then $g(n) \geq e \cdot l(n)$ where $l(n) =$ # of arcs in the path from $S$ to $n$ found so far. If A* does not terminate, $g(n)$ and hence $f(n) = g(n) + h(n)$ [$h(n) \geq 0$] will become unbounded.

This is not consistent with the lemma. So A* has to terminate.
2\textsuperscript{nd} part of admissibility of A*

The path formed by A* is optimal when it has terminated

Proof
Suppose the path formed is not optimal
Let $G$ be expanded in a non-optimal path.
At the point of expansion of $G$,

$$f(G) = g(G) + h(G)$$
$$= g(G) + 0$$
$$> g^*(G) = g^*(S) + h^*(S)$$
$$= f^*(S) \quad [f^*(S) = \text{cost of optimal path}]$$

This is a contradiction
So path should be optimal
Theorem

A version $A_2^*$ of A* that has a “better” heuristic than another version $A_1^*$ of A* performs at least “as well as” $A_1^*$

Meaning of “better”
$h_2(n) > h_1(n)$ for all $n$

Meaning of “as well as”
$A_1^*$ expands at least all the nodes of $A_2^*$

For all nodes $n$, except the goal node
Proof by induction on the search tree of $A_2^*$. 

$A^*$ on termination carves out a tree out of $G$

**Induction**
on the depth $k$ of the search tree of $A_2^*$. $A_1^*$ before termination expands all the nodes of depth $k$ in the search tree of $A_2^*$.

$k=0$. True since start node $S$ is expanded by both

Suppose $A_1^*$ terminates without expanding a node $n$ at depth $(k+1)$ of $A_2^*$ search tree.

Since $A_1^*$ has seen all the parents of $n$ seen by $A_2^*$

$$g_1(n) \leq g_2(n) \quad (1)$$
Since $A_1^*$ has terminated without expanding $n$, 
\[ f_1(n) \geq f^*(S) \] (2)

Any node whose $f$ value is strictly less than $f^*(S)$ has to be expanded.

Since $A_2^*$ has expanded $n$
\[ f_2(n) \leq f^*(S) \] (3)

From (1), (2), and (3)
\[ h_1(n) \geq h_2(n) \] which is a contradiction. Therefore, $A_1^*$ has to expand all nodes that $A_2^*$ has expanded.

**Exercise**

If better means $h_2(n) > h_1(n)$ for some $n$ and $h_2(n) = h_1(n)$ for others, then Can you prove the result?