Lecture 10: $Ax \leq b$ as a convex combination of its extreme points

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In this lecture, we complete the proof of a theorem stating that all points in the set $Ax \leq \mathbf{b}$ can be expressed as a convex combination of its extreme points. We then prove that a linear function on such a set is maximized at an extreme point, and show how that is used to construct the Simplex algorithm.

Theorem 1 *Let $p_1, p_2, p_3, \ldots, p_t$ be the extreme points of the convex set $S = \{x : Ax \leq \mathbf{b}\}$ Then every point in S can be represented as $\sum_{i=1}^{t} \lambda_i p_i$, where $\sum_{i=1}^{t} \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$*

Proof: Proof is by induction on the dimension.
Consider $p \in S$. Join p_1 to p and extend to meet q on the boundary. For the point q, we must then have

$$A_1 q = b_1 \tag{1}$$

$$A^{''} q < \mathbf{b}^{''} \tag{2}$$

(where $A^{''}$ is the rest of A), because q is on the boundary, and $A_1 q > b_1$ outside the feasible region (having crossed the hyperplane). From the first equality, we solve for one variable, say x_n and replace it throughout in $A^{''}$. This allows us to construct a new convex set $S^{'} = \{x : Cx \leq \mathbf{d}\}$, in *one less dimension.*

By the induction hypothesis, q can be written as a convex combination of extreme points in this object, $S^{'}$. Hence,

$$p = \beta p_1 + (1-\beta) q \tag{3}$$

$$= \beta p_1 + (1-\beta) \sum_{i=1}^{t^{'}} \gamma_i q_i \tag{4}$$

This however is in terms of the extreme points $q_1, q_2, q_3, \ldots, q_{t^{'}}$ of $S^{'}$. We show that the extreme points of $S^{'}$ are also extreme points of S. Suppose they were not. Let $p^{'}$ be an extreme point of $S^{'}$ but not of S. Then $\exists\ p_1^{'}, p_2^{'} \in S$ such that $p^{'} = \lambda p_1^{'} + (1-\lambda) p_2^{'}$. By construction of points in $S^{'}$,

$$b_1 = A_1 p^{'} \tag{5}$$

$$= \lambda A_1 p_1^{'} + (1-\lambda) A_1 p_2^{'} \tag{6}$$

But since we have $A_1 p_1^{'} \leq b_1$ and $A_1 p_2^{'} \leq b_1$ (both p_1 and p_2 are in S), we must have the equality holding in both for the above equality (6) to be true. Therefore $p_1^{'}$ and $p_2^{'}$ must also be in $S^{'}$. $p^{'}$ can not then be extreme in $S^{'}$, as it is the convex combination of two points in the same set.

For the base case, take the dimension to be 0. This completes the proof. □

The following theorem will put the last step in place to construct an algorithm for solving LP problems.

Theorem 2 *A linear function on $S = \{x : Ax \leq \mathbf{b}\}$ is maximized at an extreme point.*

Proof: Let a linear function f attain its maximum at point p, where $p = \sum_{i=1}^{t} \lambda_i p_i$ (This is a valid assumption by the previous theorem). Then $f(p) = \sum_{i=1}^{t} \lambda_i f(p_i)$. If all of the $f(p_i)$'s were lesser than $f(p)$, their convex combination cannot sum to $f(p)$. Therefore for at least one i, $f(p_i) = f(p)$. □

After having proved this, we have a finite algorithm at our disposal now. An extreme point is an intersection of n linearly independent hyperplanes. We just need to pick all combinations of n rows from A ($\binom{m}{n}$ in number), solve for x_0 in $A^{'}x_0 = \mathbf{b'}$ using Gaussian Elimination, **verify** that the solution indeed satisfies all other inequalities, and then calculate $c^T x$.

The verification part is important, as the n hyperplanes we choose may end up defining an infeasible point. An example is 2-D is shown in Figure 1.

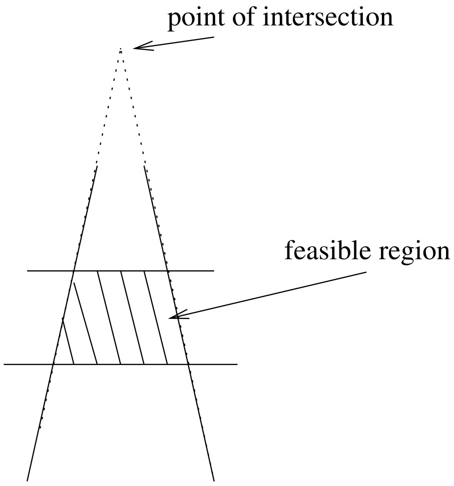

Figure 1: Why we need to verify

A rather simple formulation of the algorithm could then be:

Start at an extreme point.
While a neighbour of higher cost exists, move to it.

Intuitively, this would work, as by a previous result a local maximum in such a problem is also a global maximum. A more formal description of the Simplex algorithm and proof of its correctness is done in subsequent lectures.
Questions raised at this juncture are:

1. How do we start the process? It is pointless to obtain all extreme points and then pick one from among them.
2. How do we move to a neighbour?
3. Why are we guaranteed that the optimal is attained when we stop?