CS 435 : Linear Optimization Fall 2006

Lecture 21: Review of graph algorithms (contd.)

Lecturer: *Sundar Vishwanathan* Scribe: *Chaitanya Gokhale*
Computer Science & Engineering Indian Institute of Technology, Bombay

1 Recap of previous lecture

Previous lecture(s) dealt with the BFS based algorithm for matching in general graphs. The following lemma was stated and proved earlier :

Lemma 1 *Let M be a matching. let B be a blossom such that the base is unmatched. Let G' be the graph obtained by shrinking B. Let M' be the edges of M outside B. There is an augumenting path in G w.r.t M iff there is an augumenting path in G' w.r.t M'.*

In this lecture, we prove a complimentary lemma which completes the proof of correctness of the algorithm to find a maximum matching in a general graph.

2 Complimentary Lemma and its proof

In order to find an augmenting path in a general graph G w.r.t. matching M, we proceed in the same way as in case of bipartite graphs :

1. Start from an unmatched vertex and do a BFS search. However, in this case we might encouter a blossom.
2. If we reach a blossom B during BFS search, then switch the matched and unmatched edges of the path till the base of the bottom, as a result of which the base becomes unmatched (as shown in fig 1). Let us call the new matching as M'.
3. Now, shrink the blossom B to a single vertex forming graph G' and matching M''. Look for an augmenting path in the new graph.
   We have an augmenting path in G' w.r.t. M'' iff there exists an augmenting path in G w.r.t. matching M.

There exists an augmenting path in G w.r.t. matching M' iff there exists an augmenting path in G' w.r.t. matching M'' (by Lemma 1 above). Hence, in order to prove the correctness of algorithm, we need to prove the first part of the algorithm regarding switching, which can be stated as :

Lemma 2 *Given a graph G and matching M and a path Q starting from an unmatched node v, to the base u of a blossom B (base u being matched). Let M' be the matching obtained on switching the matched and unmatched edges along path Q. Then an augmenting path exists in G w.r.t matching M iff there exists one in G w.r.t. matching M'.*

Proof:

Suppose that there exists an augmenting path P in the Graph G, after switching i.e. w.r.t. matching M'. We have two cases to consider :

(a) The augmenting path P has no edge common with path Q :
Then the same path exists in G w.r.t. matching M, as well.

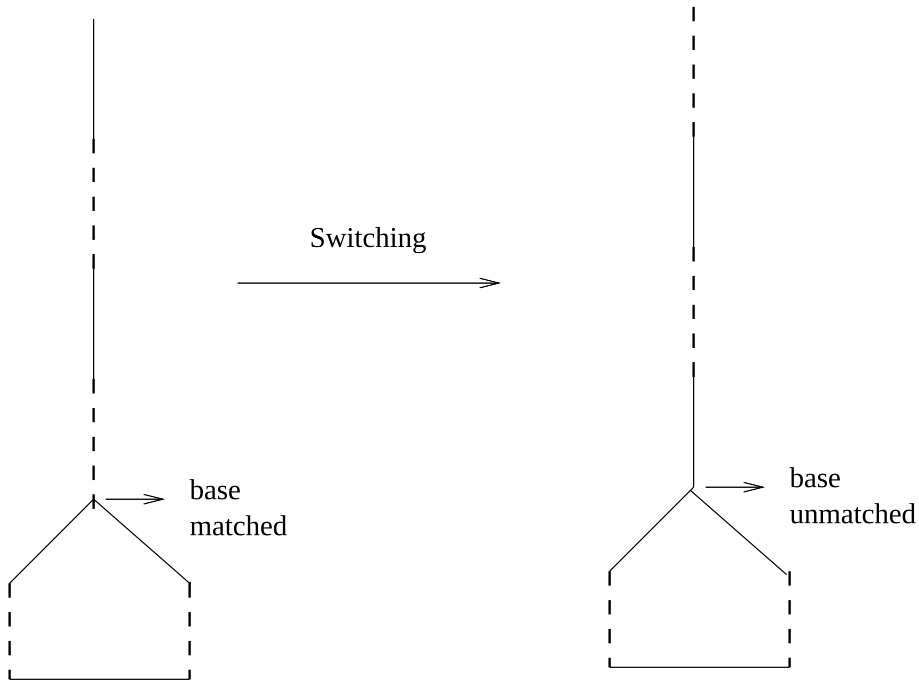

Figure 1: Switching matched and unmatched edges

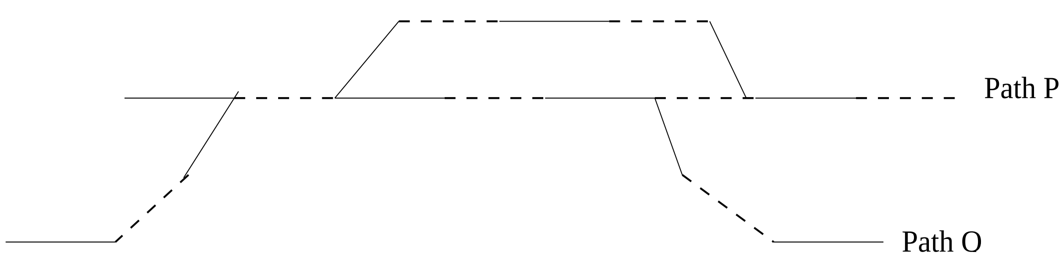

Figure 2: P & Q having Common edges

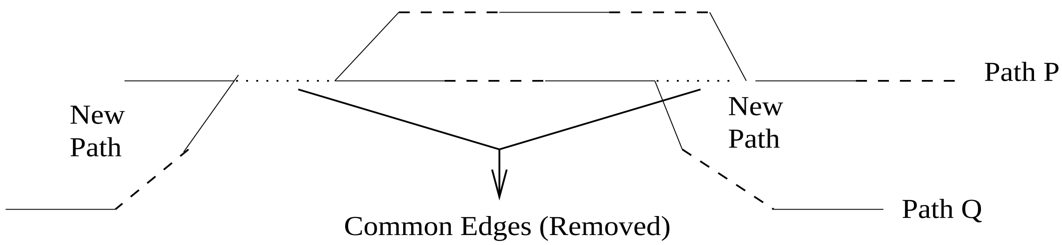

Figure 3: New paths connectign end-points after ignoring common edges

(b) The augmenting path P has some edges common with the path Q (see figure 2):

Then, we mark out the common edges to both the paths. We wish to construct an augmenting path in G w.r.t. M. For that, ignore the common edges in P and Q. We have four end points of the two paths P and Q that orginally existed. Since P is an augmenting path both its end-points are free(unmatched), while one of the end-points of Q is free. Thus, 3 of the 4 end-points are unmatched. On removing the common edges, the 2 pairs of end-points will still be connected by two different paths (refer figure 3. Since 3 of the 4 end-points are unmatched, there must be one of the two new paths having unmatched end-points. This path is alternating and has unmatched end-points. $\Rightarrow$ This is the augmenting path we are looking for.

From (a) & (b) above, if G contains an augmenting path w.r.t. $M^{\text{`}}$ then there is an augmenting path w.r.t. M as well. The above argument (cases (a) & (b)) holds for the converse as well and hence, the converse can be proved in the exact same manner.

Hence, the lemma stated above is proved.

$\square$