PAGES 216 TO 231 OF http://www.cse.iitb.ac.in/~ cs709/notes/BasicsOfConvexOptimization.pdf, interspersed with pages between 239 and 253 and summary of material thereafter, which extend univariate concepts to generic spaces

Maximum and Minimum values of univariate functions

Let f be a function with domain $\mathcal{D}$. Then f has an *absolute maximum* (or global maximum) value at point $c \in \mathcal{D}$ if

$$f(x) \leq f(c),\ \forall x \in \mathcal{D}$$

and an *absolute minimum* (or global minimum) value at $c \in \mathcal{D}$ if

$$f(x) \geq f(c),\ \forall x \in \mathcal{D}$$

If there is an open interval $\mathcal{I}$ containing c in which $f(c) \geq f(x),\ \forall x \in \mathcal{I}$, then we say that $f(c)$ is a *local maximum value* of f. On the other hand, if there is an open interval $\mathcal{I}$ containing c in which $f(c) \leq f(x),\ \forall x \in \mathcal{I}$, then we say that $f(c)$ is a *local minimum value* of f. If $f(c)$ is either a local maximum or local minimum value of f in an open interval $\mathcal{I}$ with $c \in \mathcal{I}$, the $f(c)$ is called a *local extreme value* of f.

Theorem 39 *If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.* → If all p.ds of f exist at $x = c \in \mathcal{D} \subseteq \mathbb{R}^n$ & if $f(c)$ is local extreme, $\nabla f(c) = 0$

Theorem 40 *A continuous function $f(x)$ on a closed and bounded interval $[a, b]$ attains a minimum value $f(c)$ for some $c \in [a, b]$ and a maximum value $f(d)$ for some $d \in [a, b]$. That is, a continuous function on a closed, bounded interval attains a minimum and a maximum value.*

Note: $[a, \infty)$ is closed but NOT bounded
So both conditions are needed

replace with sets for $\mathbb{R}^n$

FOR $\mathbb{R}^n$

Theorem 60 *If $f(\mathbf{x})$ defined on a domain $\mathcal{D} \subseteq \Re^n$ has a local maximum or minimum at $\mathbf{x}^*$ and if the first-order partial derivatives exist at $\mathbf{x}^*$, then $f_{x_i}(\mathbf{x}^*) = 0$ for all $1 \le i \le n$.* ie $\nabla f(x^*) = 0$

Definition 27 [Critical point]: *A point $\mathbf{x}^*$ is called a critical point of a function $f(\mathbf{x})$ defined on $\mathcal{D} \subseteq \Re^n$ if*

1. *If $f_{x_i}(\mathbf{x}^*) = 0$, for $1 \le i \le n$.*
2. *OR $f_{x_i}(\mathbf{x}^*)$ fails to exist for any $1 \le i \le n$.*

A procedure for computing all critical points of a function f is:

1. Compute f_{x_i} for $1 \le i \le n$.
2. Determine if there are any points where any one of f_{x_i} fails to exist. Add such points (if any) to the list of critical points.
3. Solve the system of equations $f_{x_i} = 0$ simultaneously. Add the solution points to the list of saddle points.

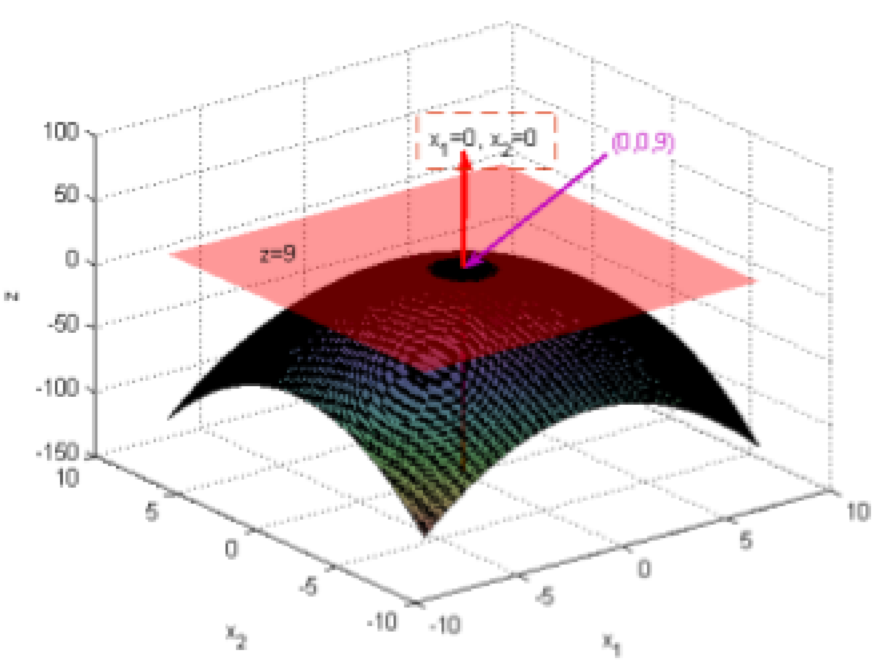


Figure 4.17: The paraboloid $f(x_1, x_2) = 9 - x_1^2 - x_2^2$ attains its maximum at $(0, 0)$. The tangent plane to the surface at $(0, 0, f(0, 0))$ is also shown, and so is

$|x_1|$

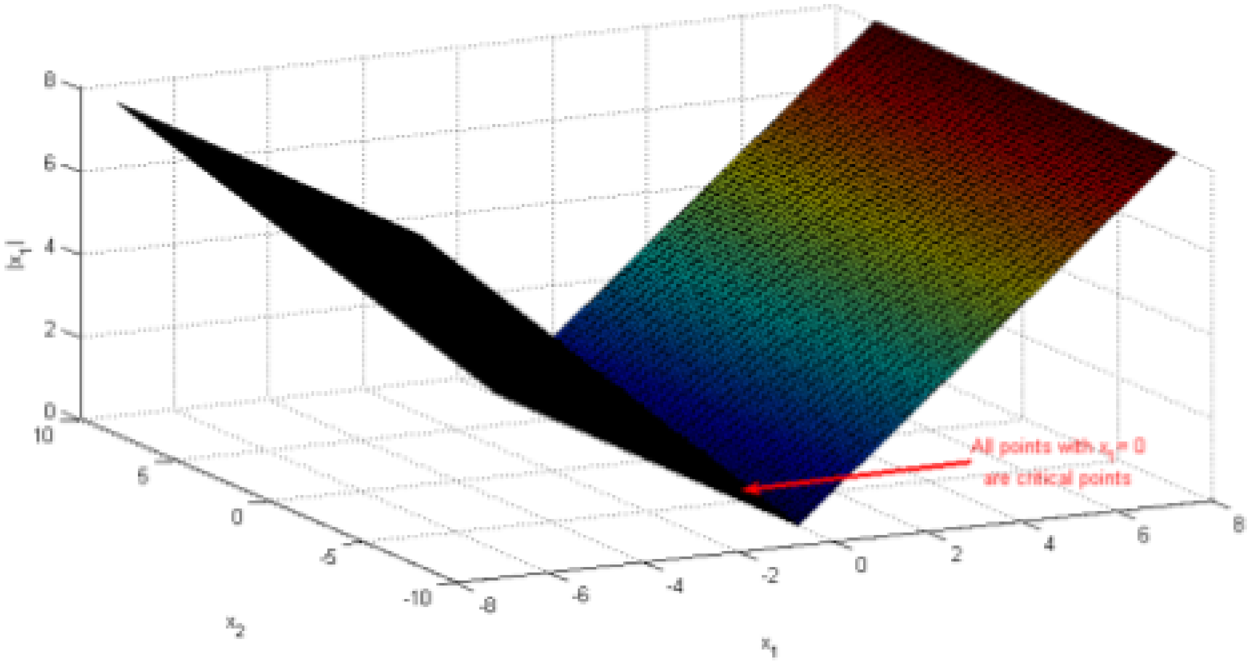


Figure 4.18: Plot illustrating critical points where derivative fails to exist.

Definition 28 [Saddle point]: *A point* $\mathbf{x}^*$ *is called a saddle point of a function* $f(\mathbf{x})$ *defined on* $\mathcal{D} \subseteq \Re^n$ *if* $\mathbf{x}^*$ *is a critical point of* f *but* $\mathbf{x}^*$ *does not correspond to a local maximum or minimum of the function.*

$x_1^2 - x_2^2$ →

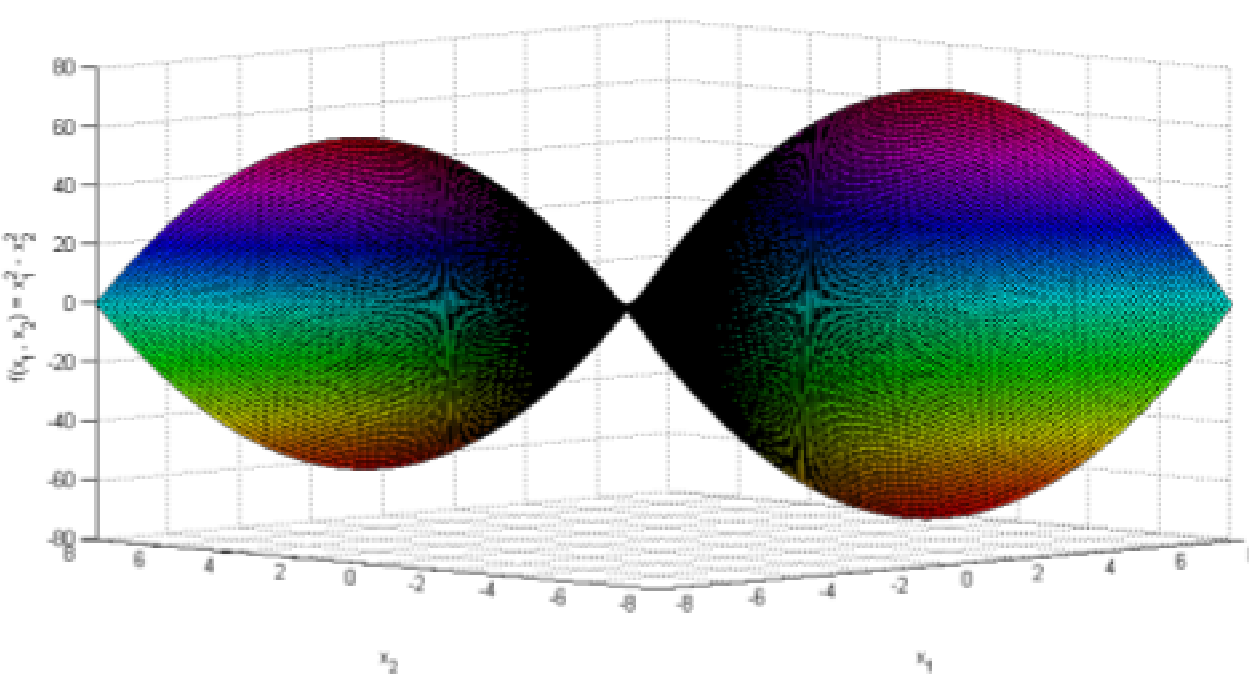


Figure 4.19: The hyperbolic paraboloid $f(x_1, x_2) = x_1^2 - x_2^2$, which has a saddle point at $(0, 0)$.

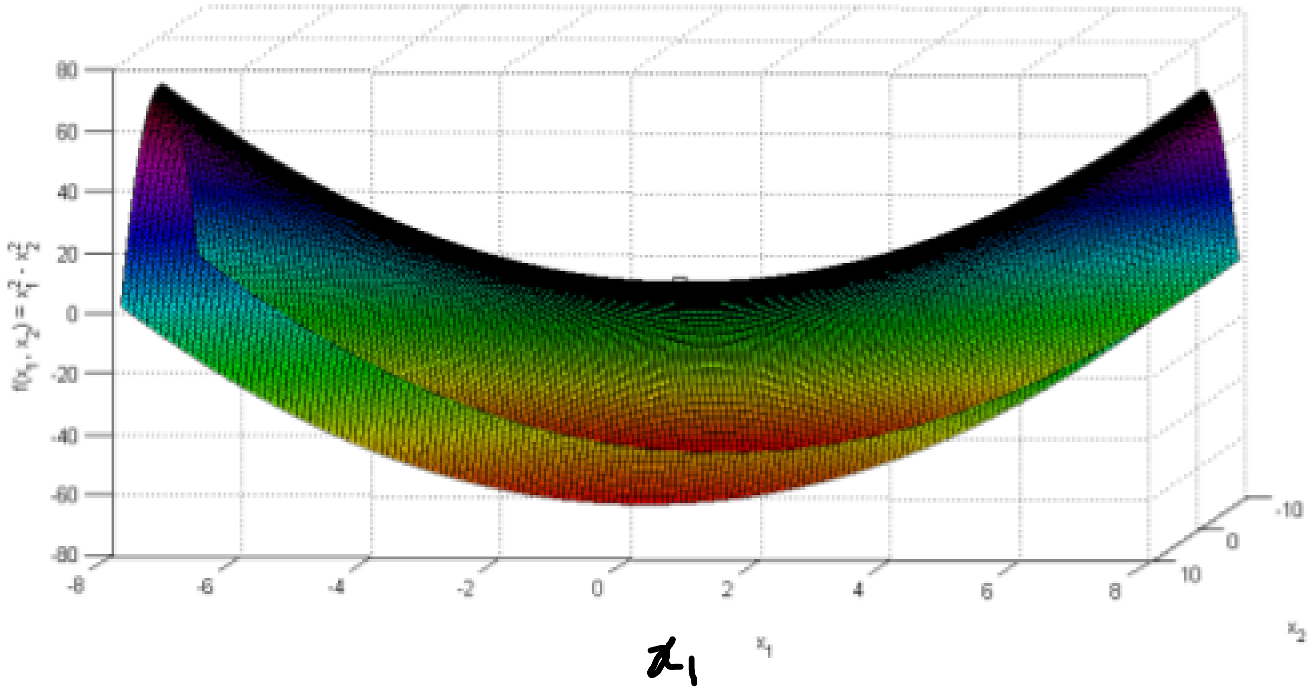


Figure 4.20: The hyperbolic paraboloid $f(x_1, x_2) = x_1^2 - x_2^2$, when viewed from the x_1 axis is concave up.

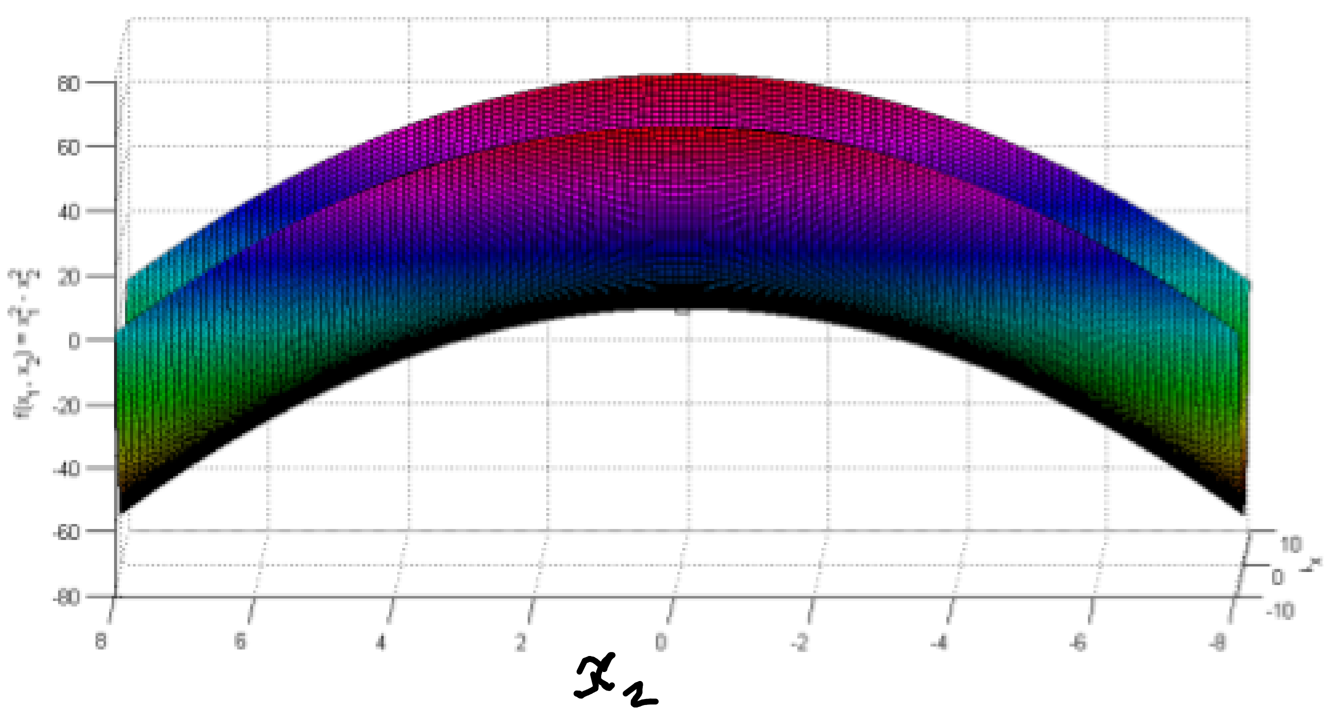
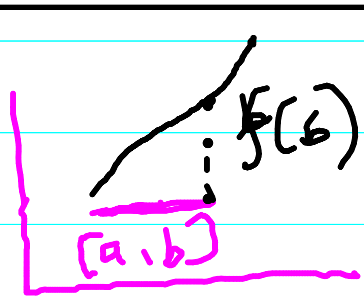


Figure 4.21: The hyperbolic paraboloid $f(x_1, x_2) = x_1^2 - x_2^2$, when viewed from the x_2 axis is concave down.



Note: For LP's, $Ax \geq b$ is closed and bounded D & $f(x) = c^T x$ attains global max/min on bdry of D. ∴ This thm not applicable

Theorem 41 *A continuous function $f(x)$ on a closed and bounded interval $[a,b]$ attains a minimum value $f(c)$ for some $c \in [a,b]$ and a maximum value $f(d)$ for some $d \in [a,b]$. If $a < c < b$ and $f'(c)$ exists, then $f'(c) = 0$. If $a < d < b$ and $f'(d)$ exists, then $f'(d) = 0$.* : If $D \subseteq \mathbb{R}^n$ is closed & bounded & f is cts on D & if global max/min is attained at $c \in \text{Int}(D)$ & f is differentiable at c then $\nabla f(c) = 0$

Theorem 42 *If f is continuous on $[a,b]$ and differentiable at all $x \in (a,b)$ and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.*

Figure 4.1 illustrates Rolle's theorem with an example function $f(x) = 9 - x^2$ on the interval $[-3,+3]$.

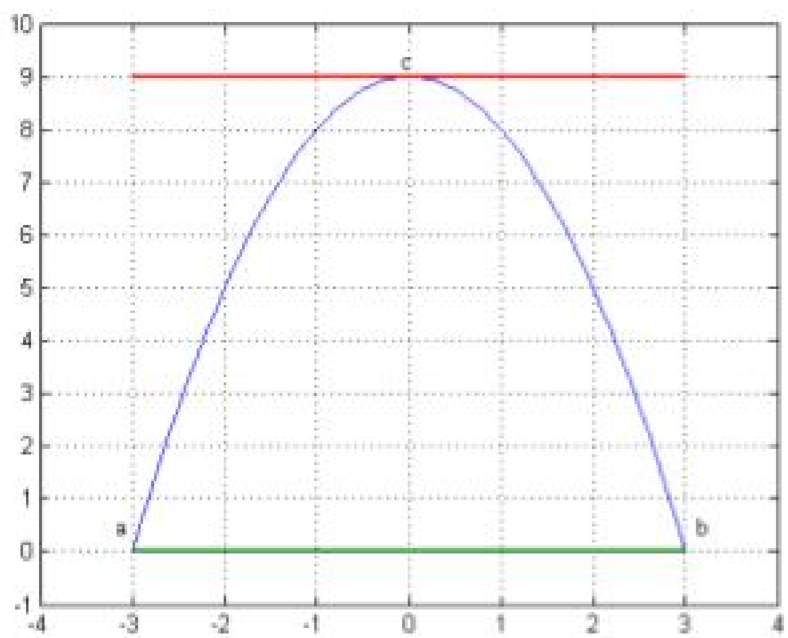


Figure 4.1: Illustration of Rolle's theorem with $f(x) = 9 - x^2$ on the interval $[-3,+3]$. We see that $f'(0) = 0$.

Q: What is a more general version of Rolle's thm?

Ans: Mean value thm

Theorem 43 *If f is continuous on $[a,b]$ and differentiable at all $x \in (a,b)$, then there is some $c \in (a,b)$ such that, $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

If $D \subseteq \mathbb{R}^n$ is closed & bounded & f is cts on D & diff on int(D) then:

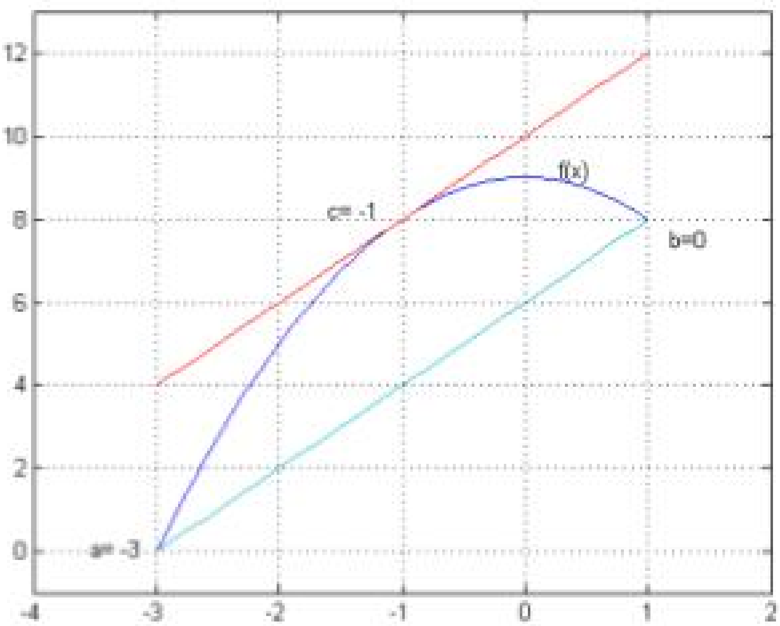


Figure 4.2: Illustration of mean value theorem with $f(x) = 9 - x^2$ on the interval $[-3, 1]$. We see that $f'(-1) = \frac{f(1)-f(-3)}{4}$.

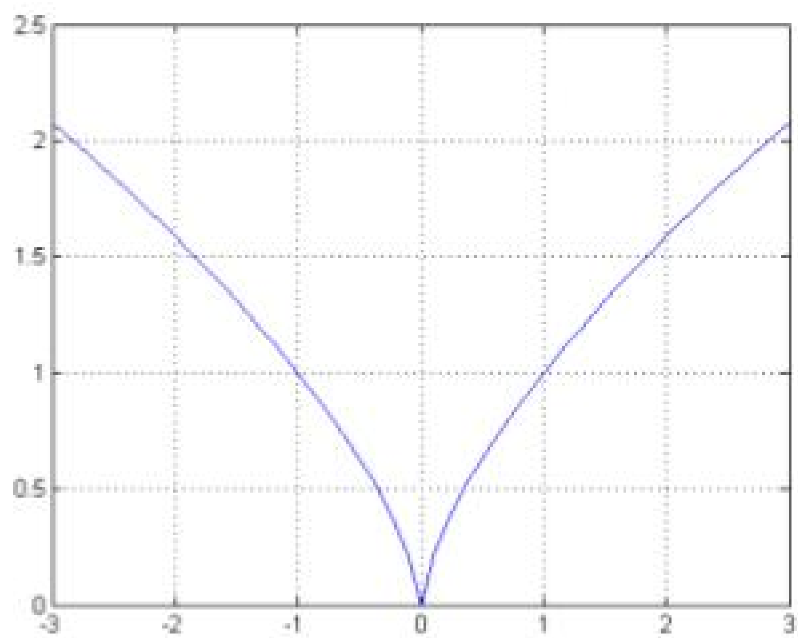


Figure 4.4: The mean value theorem can be violated if $f(x)$ is not differentiable at even a single point of the interval. Illustration on $f(x) = x^{2/3}$ with the