Developing Tools for Convexity Analysis of $f(x_1, x_2, ..x_n)$

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Local Extrema for $f(x_1, x_2.., x_n)$

Definition

[Local minimum]: *A function* $f : \mathcal{D} \to \Re$ *of* n *variables has a local minimum at* $\mathbf{x}^0$ *if* $\exists \mathcal{N}(\mathbf{x}^0)$ *such that* $\forall\ \mathbf{x} \in \mathcal{N}(\mathbf{x}^0)$, $f(\mathbf{x}^0) \leq f(\mathbf{x})$. *In other words,* $f(\mathbf{x}^0) \leq f(\mathbf{x})$ *whenever* $\mathbf{x}$ *lies in some neighborhood around* $\mathbf{x}^0$. *An example neighborhood is the circular disc when* $\mathcal{D} = \Re^n$.

Definition

[Local maximum]: .... $f(\mathbf{x}^0) \geq f(\mathbf{x})$.

General Reference: Stories About Maxima and Minima (Mathematical World) by Vladimir M. Tikhomirov

Local Extrema

These definitions are exactly analogous to the definitions for a function of single variable. Figure 7 shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

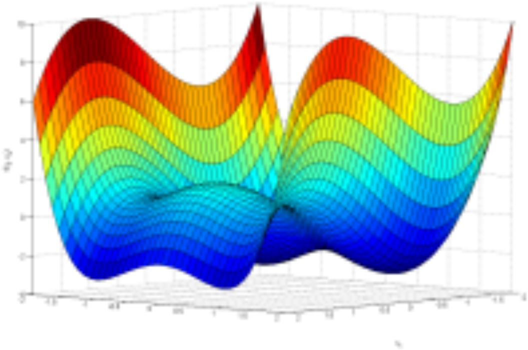

Figure 1:

Convexity and Extremum: Slopeless interpretation (SI)

Definition

A function f is convex on $\mathcal{D}$, *iff*

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \quad (1)$$

and is strictly convex on $\mathcal{D}$, *iff*

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) < \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \quad (2)$$

whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $\mathbf{x}_1 \neq \mathbf{x}_2$ and $0 < \alpha < 1$.

Note: This implicitly assumes that whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$,

Convexity and Extremum: Slopeless interpretation (SI)

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Note: This implicitly assumes that whenever $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, $\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in \mathcal{D}$

Local Extrema

Figure 2 shows the plot of $f(x_1, x_2) = 3x_1^2 + 3x_2^2 - 9$. As can be seen in the plot, the function is cup shaped and appears to be convex everywhere in $\Re^2$.

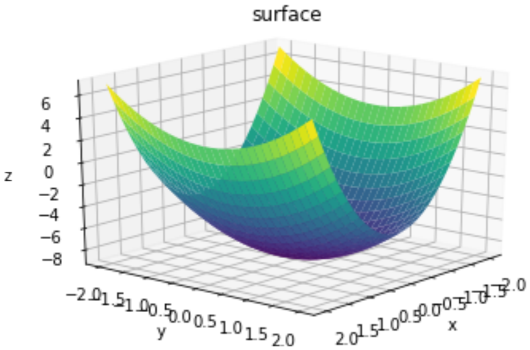


Figure 2:

From $f(x) : \Re \rightarrow \Re$ to $f(x_1, x_2 \ldots x_n) : \mathcal{D} \rightarrow \Re$

Need to also extend

- Extreme Value Theorem
- Rolle's theorem, Mean Value Theorem, Taylor Expansion
- Necessary and Sufficient first and second order conditions for local/extrema
- First and second order conditions for Convexity

Need following notions/definitions in $\mathcal{D}$

- Neighborhood and open sets/balls ($\Leftarrow$ Local extremum)
- Bounded, Closed Sets ($\Leftarrow$ Extreme value theorem)
- Convex Sets ($\Leftarrow$ Convex functions of n variables)
- Directional Derivatives and Gradients ($\Leftarrow$ Taylor Expansion, all first order conditions)

Convex Functions, Epigraphs, Sublevel sets, Separating and Supporting Hyperplane Theorems and required tools

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

Recall:

Strict Convexity and Extremum: Slopeless interpretation (SI)

Claim

A function f is strictly convex on an open interval $\mathcal{I}$, iff

$$f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2) \tag{1}$$

whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ *and* $0 < a < 1$.

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

- A function $f : \mathcal{D} \rightarrow \Re$ is **convex** if D is convex and f at convex combination is upper bounded by convex combinations of the f

Convex Functions: Extending Slopeless Definition from $\Re :\rightarrow \Re$

- A function $f: \mathcal{D} \rightarrow \Re$ is **convex** if $\mathcal{D}$ is a convex set and

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y}) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{40}$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **strictly convex** if $\mathcal{D}$ is convex and

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) < \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y})) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1 \tag{41}$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **strongly convex** if $\mathcal{D}$ is convex and for some constant $c > 0$

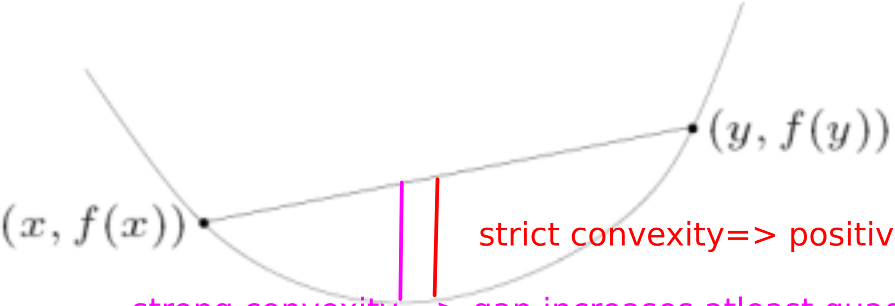
guaranteed upperbound increasing quadratically wrt ||x-y||

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y})) - \frac{1}{2}c\theta(1-\theta)||\mathbf{x}-\mathbf{y}||^2 \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1$$

- A function $f: \mathcal{D} \rightarrow \Re$ is **uniformly convex** wrt function $c(\mathbf{x}) \geq 0$ (vanishing only at 0) if $\mathcal{D}$ is convex and

c is some non-negative function of ||x-y||

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{y})) - c(\|\mathbf{x}-\mathbf{y}\|)\theta(1-\theta) \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{D} \quad 0 \leq \theta \leq 1$$



strict convexity=> positive gap

strong convexity => gap increases atleast quadratically wrt ||x-y|| (further away is x from y, weaker the approximation)

c is the strength of the convexity

Figure 5: Example of convex function.

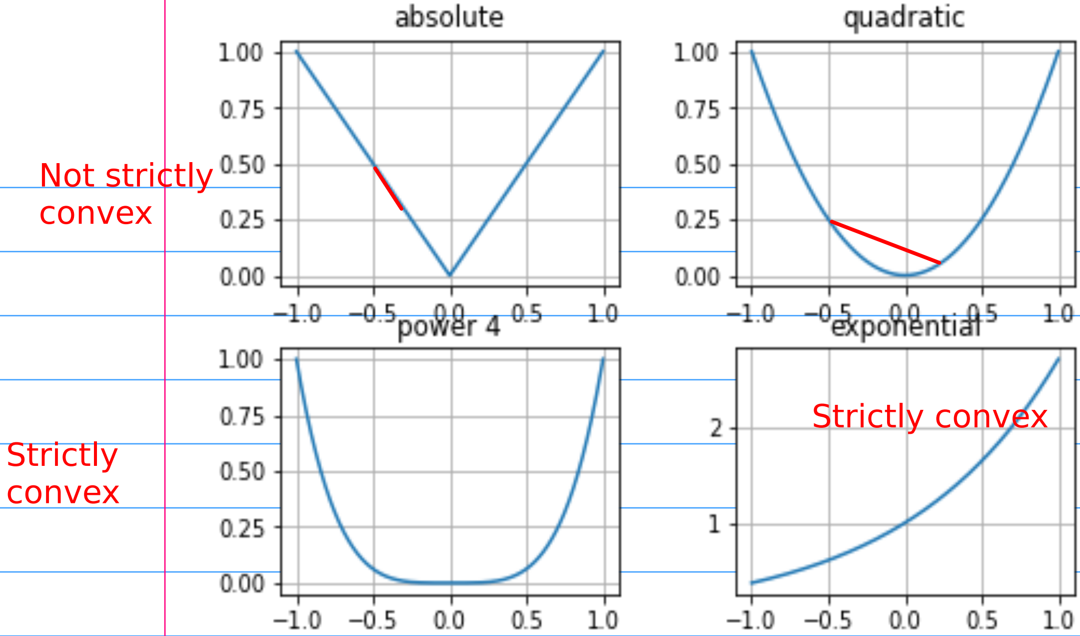
uniform convexity ==> weakness is characterized by a fixed function c(||x-y||)

Examples of Convex Functions

Examples of convex functions on the set of reals $\Re$ as well as on $\Re^n$ and $\Re^{m\times n}$ are shown below.

Function type	Domain	Additional Constraints
The affine function: $ax+b$	$\Re$	Any $a,b\in\Re$
The exponential function: e^{ax}	$\Re$	Any $a\in\Re$
Powers: x^α	$\Re_{++}$	$\alpha\geq 1$ or $\alpha\leq 1$
Powers of absolute value: $\lvert x\rvert^p$	$\Re$	$p\geq 1$
Negative entropy: $x\log x$	$\Re_{++}$	
Affine functions of vectors: $\mathbf{a}^T\mathbf{x}+b$	$\Re^n$	
p-norms of vectors: $\lvert\lvert\mathbf{x}\rvert\rvert_p=\left(\sum_{i=1}^{n}\lvert x_i\rvert^p\right)^{1/p}$	$\Re^n$	$p\geq 1$
inf norms of vectors: $\lvert\lvert\mathbf{x}\rvert\rvert_\infty=\max_k\lvert x_k\rvert$	$\Re^n$	
Affine functions of matrices: $tr(A^TX)+b=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}X_{ij}+b$	$\Re^{m\times n}$	
Spectral (maximum singular value) matrix norm: $\lvert\lvert X\rvert\rvert_2=\sigma_{max}(X)=(\lambda_{max}(X^TX))^{1/2}$	$\Re^{m\times n}$	

Table 1: Examples of convex functions on $\Re$, $\Re^n$ and $\Re^{m\times n}$.



absolute
quadratic
power 4
exponential
1.00
0.75
0.50
0.25
0.00
−1.0
−0.5
0.0
0.5
1.0
1
2
Not strictly convex
Strictly convex
Strictly convex

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:** x^2
  ax^2 + bx + c

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$,

||x||^2
||A||_F (Frobeinius norm)
x^TAx + bx + c

Strict, Strong and Uniform Convexity for $f : \Re \to \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \to \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**

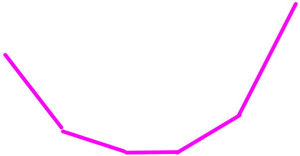
x^4
x^6
exp(x)

Strict, Strong and Uniform Convexity for $f : \Re \rightarrow \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \rightarrow \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^4$
- **Convex but not Strictly Convex:**

|x|

piecewise linear functions



Strict, Strong and Uniform Convexity for $f : \Re \rightarrow \Re$

- **Strictly, Strongly Convex Function:**
  - $f(x) = x^2$
  - $f(x) = x^2 - cos(x)$
  - For $f : \Re^n \rightarrow \Re$, $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$
- **Strictly Convex but not Strongly Convex:**
  - $f(x) = x^4$
  - $f(x) = x^4$
- **Convex but not Strictly Convex:**
  - $f(x) = |x|$

Note: Domain D of a concave function should still remain convex

A function $f : \Re^n \to \Re$ is said to be concave if the function $-f$ is convex. Examples of concave functions on the set of reals $\Re$ are shown below. If a function is both convex and concave, it must be affine, as can be seen in the two tables. H/W: Can you prove this?

Function type	**Domain**	**Additional Constraints**
The affine function: $ax + b$	$\Re$	Any $a, b \in \Re$
Powers: x^α	$\Re_{++}$	$0 \leq \alpha \leq 1$
logarithm: $\log x$	$\Re_{++}$	

Table 2: Examples of concave functions on $\Re$.

All properties we discuss for min of convex fns should hold for max of concave functions

Convexity and Global Minimum

Fundamental chracteristics:

1. Any point of local minimum point is also a point of global minimum.
2. For any stricly convex function, the point corresponding to the gobal minimum is also unique.

To discuss these further, we need to extend the defitions of Local Minima/Maxima to arbitrary sets $\mathcal{D}$

Illustrating Local Extrema for $f: \Re^2 \rightarrow \Re$

These definitions are exactly analogous to the definitions for a function of single variable. Figure below shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

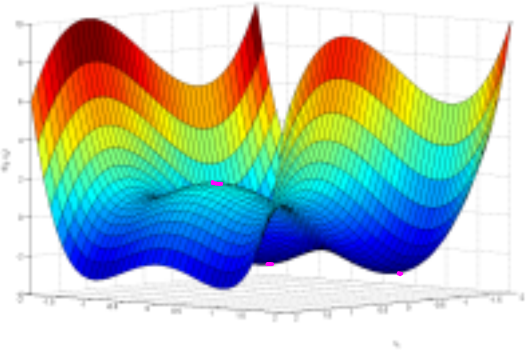


Figure 6:

Local Extrema in Normed Spaces: Extending from $\Re \to \Re$

Recap:

Let $f : \mathcal{D} \to \Re$. Now f has

- An *absolute maximum* (or global maximum) value at point $c \in \mathcal{D}$ if
$$f(x) \leq f(c),\ \forall x \in \mathcal{D}$$
- An *absolute minimum* (or global minimum) value at $c \in \mathcal{D}$ if
$$f(x) \geq f(c),\ \forall x \in \mathcal{D}$$
- A *local maximum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \geq f(x),\ \forall x \in \mathcal{I}$
- A *local minimum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \leq f(x),\ \forall x \in \mathcal{I}$
- A *local extreme value* at c, if $f(c)$ is either a local maximum or local minimum value of f in an open interval $\mathcal{I}$ with $c \in \mathcal{I}$

Local Extrema in Normed Spaces: Extending from $\Re \rightarrow \Re$

Definition

[Local maximum]: *A function f of n variables has a local maximum at* $\mathbf{x}^0 \in \mathcal{D}$ *in a normed space* $\mathcal{D}$ *if* $\exists \epsilon > 0$ *such that* $\forall\ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon.\ f(\mathbf{x}) \leq f(\mathbf{x}^0)$. *In other words,* $f(\mathbf{x}) \leq f(\mathbf{x}^0)$ *whenever* $\mathbf{x}$ *lies in the interior of some norm ball around* $\mathbf{x}^0$.

Definition

[Local minimum]: *A function f of n variables has a local minimum at* $\mathbf{x}^0 \in \mathcal{D}$ *in a normed space* $\mathcal{D}$ *if* $\exists \epsilon > 0$ *such that* $\forall\ ||\mathbf{x} - \mathbf{x}^0|| < \epsilon.\ f(\mathbf{x}) \geq f(\mathbf{x}^0)$. *In other words,* $f(\mathbf{x}) \geq f(\mathbf{x}^0)$ *whenever* $\mathbf{x}$ *lies in the interior of some norm ball around* $\mathbf{x}^0$.

1. These definitions can be easily extended to metric spaces or topological spaces. But we need definitions of open sets and interior in those spaces (and in fact some other foundations will also help).
2. We will first provide these defintions in $\Re^n$ and then provide the idea for extending them to more abstract topological/metric/normed spaces.

Recap: Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Balls in $\Re^n$]: Consider a point $\mathbf{x} \in \Re^n$. Then the closed norm ball around $\mathbf{x}$ of radius ϵ is

$$\mathcal{B}[\mathbf{x}, \epsilon] = \{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| \leq \epsilon\}$$

Likewise, the open nborm all around $\mathbf{x}$ of radius ϵ is defined as

$$\mathcal{B}(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \Re^n | ||\mathbf{y} - \mathbf{x}|| < \epsilon\}$$

Open norm ball = int(closed norm ball)

For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there is a closed ball B[x,\epsilon] containing S

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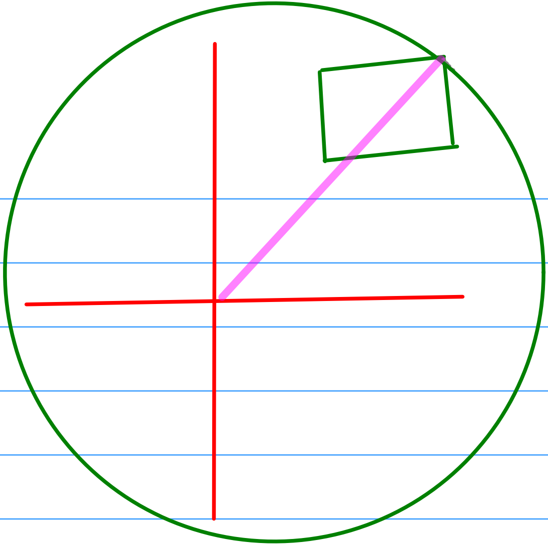
For the 1-D case, open and closed balls degenerate to open and closed intervals respectively.

Definition

[Boundedness in $\Re^n$]: We say that a set $\mathcal{S} \subset \Re^n$ is *bounded* when there exists an $\epsilon > 0$ such that $\mathcal{S} \subseteq \mathcal{B}[0, \epsilon]$. Note that the centre of the closed ball is 0

In other words, a set $\mathcal{S} \subseteq \Re^n$ is bounded means that there exists a number $\epsilon > 0$ such that for all $\mathbf{x} \in \mathcal{S}$, $||\mathbf{x}|| \leq \epsilon$. Eg: the positive quadrant is not bounded. But any rectangle is bounde



Interpretation of bounded set (the rectangle in green is bounded by the circle in green)

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if

there exists an open call B(x,\epsilon) contained in S

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points that are interior points

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Interior and Boundary points]: A point $\mathbf{x}$ is called an *interior point* of a set $\mathcal{S}$ if there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subseteq \mathcal{S}$.

In other words, a point $\mathbf{x} \in \mathcal{S}$ is called an interior point of a set $\mathcal{S}$ if there exists an open ball of non-zero radius around $\mathbf{x}$ such that the ball is completely contained within $\mathcal{S}$.

Definition

[Interior of a set]: Let $\mathcal{S} \subseteq \Re^n$. The set of all points lying in the interior of $\mathcal{S}$ is denoted by $int(\mathcal{S})$ and is called the *interior* of $\mathcal{S}$. That is,

$$int(\mathcal{S}) = \left\{\mathbf{x} | \exists \epsilon > 0 \ s.t. \ \mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}\right\}$$

In the $1-$D case, the open interval obtained by excluding endpoints from an interval $\mathcal{I}$ is the interior of $\mathcal{I}$, denoted by $int(\mathcal{I})$. For example, $int([a, b]) = (a, b)$ and $int([0, \infty)) = (0, \infty)$.

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

1) If the boundary were to belong to S:
set of all points x such that any open ball (with any epsilon) around x is partly outside S (and implicitly partly contained in S)
2) In general:
set of all points x such that any open ball (with any epsilon) around x is partly outside S and partly contained in S

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

$$\partial(\mathcal{S}) = \left\{ \mathbf{y} | \forall\ \epsilon > 0,\ \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset \textit{ and } \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S}^C \neq \emptyset \right\}$$

part of ball in the set

part of ball outside (in the complen

For example, *partial*$([a, b]) = \{a, b\}$.

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when,

the boundary does not belong to the set

More Basic Prerequisite Topological Concepts in $\Re^n$

Definition

[Boundary of a set]: Let $\mathcal{S} \subseteq \Re^n$. The boundary of $\mathcal{S}$, denoted by $\partial(\mathcal{S})$ is defined as

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For example, $partial([a, b]) = \{a, b\}$.

Definition

[Open Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is an *open set* when, for every $\mathbf{x} \in \mathcal{S}$, there exists an $\epsilon > 0$ such that $\mathcal{B}(\mathbf{x}, \epsilon) \subset \mathcal{S}$.

1. The simplest examples of an open set are the open ball, the empty set $\emptyset$ and $\Re^n$.
2. Further, arbitrary union of opens sets is open. Also, finite intersection of open sets is open. Recall Topology: Basic entry point of open sets
3. The interior of any set is always open. It can be proved that a set $\mathcal{S}$ is open if and only if $int(\mathcal{S}) = \mathcal{S}$.

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when

its complement is open

More Basic Prerequisite Topological Concepts in $\Re^n$

The complement of an open set is the closed set.

Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

The closed ball, the empty set $\emptyset$ and $\Re^n$ are three simple examples of closed sets. Arbitrary intersection of closed sets is closed. Furthermore, finite union of closed sets is closed.

Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by $closure(\mathcal{S})$ is given by

union of S with its boundary

More Basic Prerequisite Topological Concepts in $\Re^n$

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Definition

[Closed Set]: Let $\mathcal{S} \subseteq \Re^n$. We say that $\mathcal{S}$ is a *closed set* when $\mathcal{S}^C$ (that is the complement of $\mathcal{S}$) is an open set. It can be proved that $\partial\mathcal{S} \subseteq \mathcal{S}$, that is, a closed set contains its boundary.

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Definition

[Closure of a Set]: Let $\mathcal{S} \subseteq \Re^n$. The closure of $\mathcal{S}$, denoted by *closure*$(\mathcal{S})$ is given by

$$\textit{closure}(\mathcal{S}) = \left\{\mathbf{y} \in \Re^n | \forall\ \epsilon > 0, \mathcal{B}(\mathbf{y}, \epsilon) \cap \mathcal{S} \neq \emptyset\right\}$$

Some more Interesting Connections

1. The closure of a set is the smallest closed set containing the set. The closure of a closed set is the set itself.
2. $\mathcal{S}$ is closed if and only if $\mathit{closure}(\mathcal{S}) = \mathcal{S}$.
3. A bounded set can be defined in terms of a closed set; A set $\mathcal{S}$ is bounded if and only if it is contained strictly inside a closed set.
4. A relationship between the interior, boundary and closure of a set $\mathcal{S}$ is $\mathit{closure}(\mathcal{S}) = \mathit{int}(\mathcal{S}) \cup \partial(\mathcal{S})$.

Extending Open, Closed sets, Boundary, Interior, etc to Topological Sets

This is for Optinal Reading

1. Recap: Open Set follows from Defintion 1 of Topology. Neighborhood follows from Definition 2 of Topology.
2. **Limit Point:** Let S be a subset of a topological set X. A point $x \in X$ is a limit point of S if every neighborhood of x contains atleast one point of S different from x itself.
   - If X has an associated metric d and $S \subseteq X$ then $x \in S$ is a limit point of S iff $\forall\ \epsilon > 0$, $\{y \in S \ s.t. \ 0 < d(y,x) < \epsilon\} \neq \emptyset\}$.
3. **Closure of S** $= closure(S) = S \cup \{\text{limit points of } S\}$.
4. **Boundary ∂S of S:** Is the subset of S such that every neighborhood of a point from ∂S contains atleast one point in S and one point not in S.
   - If S has a metric d then:
     $\partial S = \{x \in S | \forall\ \epsilon > 0, \exists\ y \ s.t. \ d(x,y) < \epsilon \ and \ y \in S \ and \exists\ z \ s.t. \ d(x,z) < \epsilon \ and \ z \notin S\}$
5. **Open set S:** Does not contain any of its boundary points
   - If X has an associated metric d and $S \subseteq X$ is called open if for any $x \in S$, $\exists\ \epsilon > 0$ such that given any $y \in S$ with $d(y,x) < \epsilon$, $y \in S$.
6. **Closed set S:** Has an open complement S^C

Revisiting Example for Local Extrema

Figure below shows the plot of $f(x_1, x_2) = 3x_1^2 - x_1^3 - 2x_2^2 + x_2^4$. As can be seen in the plot, the function has several local maxima and minima.

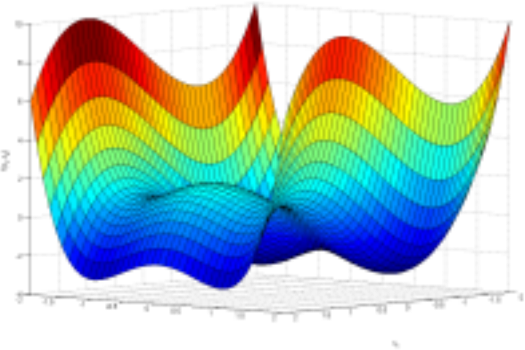

Figure 7:

Convexity and Global Minimum

Fundamental chracteristics: **Let us now prove them**

1. Any point of local minimum point is also a point of global minimum.
2. For any stricly convex function, the point corresponding to the gobal minimum is also unique.