Epigraph, Convexity, Gradients and Level-sets [OPTIONAL]

- **Revisiting level sets:** We can embed the graph of a function of n variables as the 0-level set of a function of $n+1$ variables
- More concretely, if $f: \mathcal{D} \rightarrow \Re,\ \mathcal{D} \subseteq \Re^n$ then we define $F: \mathcal{D}' \rightarrow \Re,\ \mathcal{D}' = \mathcal{D} \times \Re$ as $F(\mathbf{x}, z) = f(\mathbf{x}) - z$ with $\mathbf{x} \in \mathcal{D}'$.
- The gradient of F at any point $(\mathbf{x}, z)$ is simply, $\nabla F(\mathbf{x}, z) = \left[f_{x_1}, f_{x_2}, \ldots, f_{x_n}, -1\right]$ with the first n components of $\nabla F(\mathbf{x}, z)$ given by the n components of $\nabla f(\mathbf{x})$.
- The graph of f can be recovered as the $0-$level set of F given by $F(\mathbf{x}, z) = 0$.
- The equation of the tangent hyperplane $(\mathbf{y}, z)$ to the $0-$level set of F at the point $(\mathbf{x}, f(\mathbf{x}))$ is[1] $\nabla^T F(\mathbf{x}, f(\mathbf{x})).[\mathbf{y} - \mathbf{x}, z - f(\mathbf{x})]^T = [\nabla f(\mathbf{x}), -1]^T.[\mathbf{y} - \mathbf{x}, z - f(\mathbf{x})]^T = 0$.

[1](that is, the tangent hyperplane to $f(\mathbf{x})$ at the point $\mathbf{x}$)

Epigraph, Convexity, Gradients and Level-sets (contd.) [OPTIONAL]

Substituting appropriate expression for $\nabla F(\mathbf{x})$, the equation of the tangent plane $(\mathbf{y}, z)$ can be written as

$$\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{x})(y_i - x_i)\right) - \big(z - f(\mathbf{x})\big) = 0$$

or equivalently as,

$$\left(\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x})\right) + f(\mathbf{x}) = z$$

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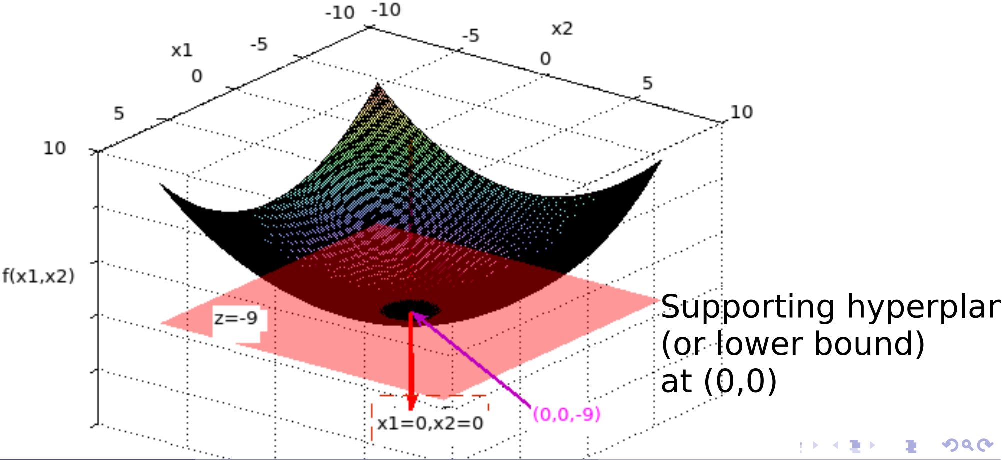
Revisiting the gradient-based condition for convexity in (8), we have that for a convex function, $f(\mathbf{y})$ is greater than each such z on the hyperplane: $f(\mathbf{y}) \geq z = f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x})$

Gradient and Epigraph (contd)

As an example, consider the paraboloid, $f(x_1, x_2) = x_1^2 + x_2^2 - 9$ that attains its minimum at $(0, 0)$. We see below its epigraph.



Illustrations to understand Gradient

- For the paraboloid, $f(x_1, x_2) = x_1^2 + x_2^2 - 9$, the corresponding $F(x_1, x_2, z) = x_1^2 + x_2^2 - 9 - z$ and the point $x^0 = (\mathbf{x}^0, z) = (1, 1, -7)$ which lies on the 0-level surface of F. The gradient $\nabla F(x_1, x_2, z)$ is $[2x_1,\ 2x_2,\ -1]$, which when evaluated at $x^0 = (1, 1, -7)$ is $[-2,\ -2,\ -1]$. The equation of the tangent plane to f at x^0 is therefore given by $2(x_1 - 1) + 2(x_2 - 1) - 7 = z$.
- The paraboloid attains its minimum at $(0, 0)$. Plot the tanget plane to the surface at $(0, 0, f(0, 0))$ as also the gradient vector ∇F at $(0, 0, f(0, 0))$. What do you expect?

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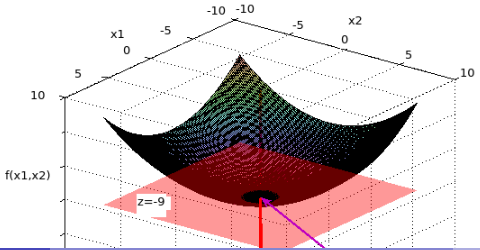
- The paraboloid attains its minimum at $(0, 0)$. Plot the tanget plane to the surface at $(0, 0, f(0, 0))$ as also the gradient vector ∇F at $(0, 0, f(0, 0))$. What do you expect? Ans: A horizontal tanget plane and a vertical gradient!



Theorem

1. *For differentiable* $f : \mathcal{D} \rightarrow \Re$ *and open convex set* $\mathcal{D}$, *f is convex* **iff**, *for any* $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \tag{9}$$

2. *f is strictly convex* **iff**, *for any* $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, *with* $\mathbf{x} \neq \mathbf{y}$,

$$f(\mathbf{y}) > f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \tag{10}$$

Strict lower bound

3. *f is strongly convex* **iff**, *for any* $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, *and for some constant* $c > 0$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2 \tag{11}$$

First-Order Convexity Condition: Proof

Proof:

Sufficiency: The proof of sufficiency is very similar for all the three statements of the theorem. So we will prove only for statement (9). Suppose (9) holds. Consider $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$ and any $\theta \in (0,1)$. Let $\mathbf{x} = \theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2$. Then,

$$\begin{aligned} f(\mathbf{x}_1) &\geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{x}_1 - \mathbf{x}) \\ f(\mathbf{x}_2) &\geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{x}_2 - \mathbf{x}) \end{aligned} \tag{12}$$

multiply by theta

multiply by 1-theta

And add..

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Adding $(1-\theta)$ times the second inequality to θ times the first, we get,

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which proves that $f(\mathbf{x})$ is a convex function. In the case of strict convexity,

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$$\theta \frac{1}{2} c||\mathbf{x} - \mathbf{x}_1||^2 + (1-\theta)\frac{1}{2}c||\mathbf{x} - \mathbf{x}_2||^2 = \frac{1}{2}c\theta(1-\theta)||\mathbf{x}_2 - \mathbf{x}_1||^2$$

First-Order Convexity Conditions: Proofs

Necessity: Suppose f is convex. Then for all $\theta \in (0,1)$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, we must have

$$f(\theta \mathbf{x}_2 + (1-\theta)\mathbf{x}_1) \leq \theta f(\mathbf{x}_2) + (1-\theta) f(\mathbf{x}_1)$$

Thus,

$\nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) =$ Directional derivative of f at x1 along x2 - x1

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This proves necessity for (9). The necessity proofs for (10) and (11) are very similar, except for a small difference for the case of strict convexity; the strict inequality is not preserved when we take limits. Suppose equality does hold in the case of strict convexity, that is for a strictly convex function f, let

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) \tag{13}$$

for some $\mathbf{x}_2 \neq \mathbf{x}_1$.

First-Order Convexity Conditions: Proofs

Necessity (contd for strict case):
Because f is stricly convex, for any $\theta \in (0,1)$ we can write

$$f\big((1-\theta)\mathbf{x}_1 + \theta\mathbf{x}_2\big) = f\big(\mathbf{x}_1 + \theta(\mathbf{x}_2 - \mathbf{x}_1)\big) < (1-\theta)f(\mathbf{x}_1) + \theta f(\mathbf{x}_2) \tag{14}$$

Since (9) is already proved for convex functions, we use it in conjunction with (13), and (14), to get

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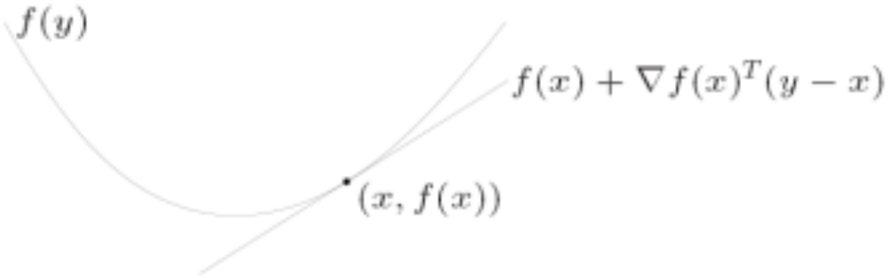
Since (9) is already proved for convex functions, we use it in conjunction with (13), and (14), to get

$$f(\mathbf{x}_1) + \theta\nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1) \leq f\big(\mathbf{x}_1 + \theta(\mathbf{x}_2 - \mathbf{x}_1)\big) < f(\mathbf{x}_1) + \theta\nabla^T f(\mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1)$$

which is a contradiction. Thus, equality can never hold in (9) for any $\mathbf{x}_1 \neq \mathbf{x}_2$. This proves the necessity of (10).

First-Order Convexity Conditions: The complete statement

The geometrical interpretation of this theorem is that at any point, the linear approximation based on a local derivative gives a lower estimate of the function, *i.e.* the convex function always lies above the supporting hyperplane at that point. This is pictorially depicted below:



(Tight) Lower-bound for any (non-differentiable) Convex Function?

For any convex function f (even if non-differentiable)

An intuitive argument though not rigourous

- The epi-graph $epi(f)$ will be convex

(Tight) Lower-bound for any (non-differentiable) Convex Function?

For any convex function f (even if non-differentiable)

- The epi-graph $epi(f)$ will be convex
- The convex epi-graph $epi(f)$ will have a supporting hyperplane at any boundary point (x,f(x))

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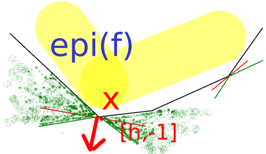

There exist multiple supporting hyperplanes

Let a supporting hyperplane be characterized by a normal vector [h(x), -1]

When f was differentiable, this vector was [gradient(x), -1]

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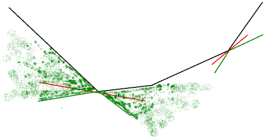

- $\left\{[\mathbf{v}, z] \mid \left\langle \left[\mathbf{h}(\mathbf{x}), -1\right], [\mathbf{v}, z]\right\rangle = \left\langle \left[\mathbf{h}(\mathbf{x}), -1\right], [\mathbf{x}, f(\mathbf{x})]\right\rangle\right\}$ for all $[\mathbf{v}, z]$ on the hyperplane and $\left\langle \left[\mathbf{h}(\mathbf{x}), -1\right], [\mathbf{y}, z]\right\rangle \leq \left\langle \left[\mathbf{h}(\mathbf{x}), -1\right], [\mathbf{x}, f(\mathbf{x})]\right\rangle$ for all $[\mathbf{y}, z] \in epi(f)$ which also includes [y,f(y)]

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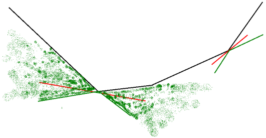

  - $\left\{[\mathbf{v}, z] \mid \langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{v}, z]\rangle = \langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{x}, f(\mathbf{x})]\rangle\right\}$ for all $[\mathbf{v}, z]$ on the hyperplane and $\langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{y}, z]\rangle \leq \langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{x}, f(\mathbf{x})]\rangle$ for all $[\mathbf{y}, z] \in epi(f)$ which also includes $[\mathbf{y}, f(\mathbf{y})]$

  Thus: $\langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{y}, f(\mathbf{y})]\rangle \leq \langle [\mathbf{h}(\mathbf{x}), -1], [\mathbf{x}, f(\mathbf{x})]\rangle$ for all $\mathbf{y} \in$ domain of f

- The normal to such a supporting hyperplane serves the same purpose as the [gradient(x),-1]

(Tight) Lower-bound for any (non-differentiable) Convex Function?

For any convex function f (even if non-differentiable)

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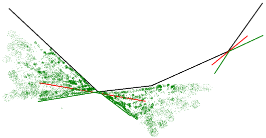

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- The normal to such a supporting hyperplane serves the same purpose as the gradient vector. It is called a **Sub-gradient** vector

The What, Why and How of (sub)gradients

1. What of (sub)gradient: Normal to supporting hyperplane at point (x,f(x) of epi(f)
   Need not be unique
   Gradient is a subgradient when the function is differentiable

The What, Why and How of (sub)gradients

1. What of (sub)gradient: Normal to the tightly lower bounding linear approximation to a convex function
2. Why of (sub)gradient: (sub)Gradient necessary and sufficient conditions of optimality for convex functions
   Important for algorithms for optimization
   Subgradients are important for non-differentiable functions
   and constraint optimization

The What, Why and How of (sub)gradients

1. What of (sub)gradient: Normal to the tightly lower bounding linear approximation to a convex function
2. Why of (sub)gradient: Ability to deal with Constraints, Optimality Conditions, Optimization Algorithms
3. How of (sub)gradient: How to compute subgradient of complex non-differentiable convex functions
   Calculus of convex functions and of subgradients

The What, Why and How of (sub)gradients

1. What of (sub)gradient: Normal to the tightly lower bounding linear approximation to a convex function
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3. How of (sub)gradient: Calculus of Convex functions and of (sub)gradients

The What of (Sub)Gradient

First-Order Convexity Conditions: Subgradients

The foregoing result motivates the definition of the *subgradient* for non-differentiable convex functions, which has properties very similar to the gradient vector.

Definition

[Subgradient]: Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a *subgradient* of f at the point $\mathbf{x} \in \mathcal{D}$ if

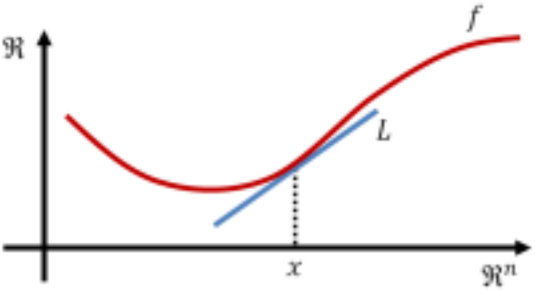
$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

For a differentiable convex function, the gradient at point $\mathbf{x}$ is the only subgradient at that point. Most properties of differentiable convex functions that hold in terms of the gradient also hold in terms of the subgradient for non-differentiable convex functions.

Eg: Subgradient for $f(\mathbf{x}) = \|\mathbf{x}\|_1$ is ? Once we develop tools (the HOW part) we will see that the subdifferential contains infinite such h at some points x

(Sub)Gradients and Convexity (contd)

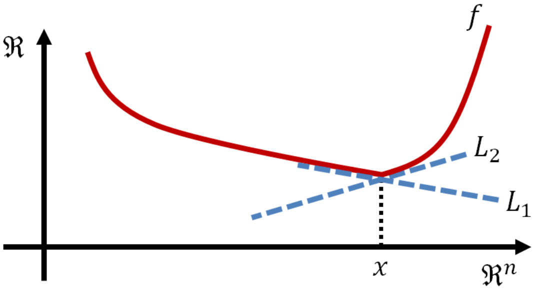


Can you think of a non-convex function f which has a non-empty subdifferential (atleast at some points x)?
Could this be for the negative of the Gaussian?

To say that a function $f : \Re^n \mapsto \Re$ is differentiable at $\mathbf{x}$ is to say that there is a (single unique) linear tangent that under estimates the function:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \triangledown f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}), \ \ \forall \mathbf{x}, \mathbf{y}$$

(Sub)Gradients and Convexity (contd)

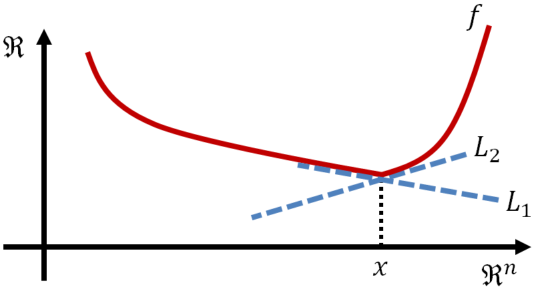


In this figure we see the function f at $\mathbf{x}$ has many possible linear tangents that may fit appropriately. Recall that a **subgradient** is any $\mathbf{h} \in \Re^n$ (same dimension as x) such that:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x}), \ \forall \mathbf{y}$$

Thus, intuitively, if a function is differentiable at a point $\mathbf{x}$ then

(Sub)Gradients and Convexity (contd)



In this figure we see the function f at $\mathbf{x}$ has many possible linear tangents that may fit appropriately. Recall that a **subgradient** is any $\mathbf{h} \in \Re^n$ (same dimension as x) such that:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x}), \;\; \forall \mathbf{y}$$

Thus, intuitively, if a function is differentiable at a point $\mathbf{x}$ then it has a unique subgradient at that point $(\nabla f(\mathbf{x}))$. Formal Proof?

(Sub)Gradients and Convexity (contd)

- A **subdifferential** is the closed convex set of all subgradients of the convex function f:

$$\partial f(\mathbf{x}) = \{\mathbf{h} \in \Re^n : \mathbf{h} \text{ is a subgradient of } f \text{ at } \mathbf{x}\}$$

  Note that this set is guaranteed to be nonempty unless f is not convex.

- Often an indicator function, $I_C : \Re^n \mapsto \Re$, is employed to remove the contraints of an optimization problem (note that convex set $C \subseteq \Re^n$):

$$\min_{\mathbf{x} \in C} f(\mathbf{x}) \iff \min_{\mathbf{x}} f(\mathbf{x}) + I_C(\mathbf{x}), \quad \text{where} \quad I_C(\mathbf{x}) = I\{\mathbf{x} \in C\} = \begin{cases} 0 & \text{if } \mathbf{x} \in C \\ \infty & \text{if } \mathbf{x} \notin C \end{cases}$$

  The subdifferential of the indicator function at x is H/W