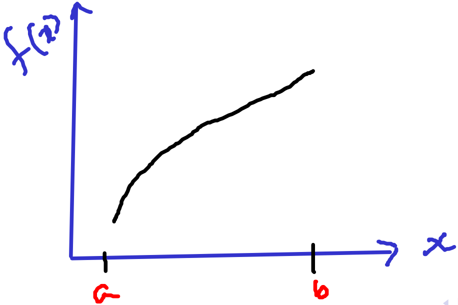
① continuity line segment ③ f differentiable on (a,b)

② Closed interval $[a,b]$

Open interval: $(a,b) = [a,b] \setminus \{a,b\}$

Optimization Principles for Univariate Functions



Maximum and Minimum values of univariate functions

$\mathcal{D}$ = union of several open/closed intervals $\subseteq \mathbb{R}$

Let $f : \mathcal{D} \rightarrow \Re$. Now f has

- An *absolute maximum* (or global maximum) value at point $c \in \mathcal{D}$ if

$$f(x) \leq f(c), \ \forall x \in \mathcal{D}$$

- An *absolute minimum* (or global minimum) value at $c \in \mathcal{D}$ if

$$f(x) \geq f(c), \ \forall x \in \mathcal{D}$$

- A *local maximum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \geq f(x), \ \forall x \in \mathcal{I}$ (p, q)
- A *local minimum value* at c if there is an open interval $\mathcal{I}$ containing c in which $f(c) \leq f(x), \ \forall x \in \mathcal{I}$
- A *local extreme value* at c, if $f(c)$ is either a local maximum or local minimum value of f in an open interval $\mathcal{I}$ with $c \in \mathcal{I}$

First Derivative Test

First derivative test for local extreme value of f, when f is differentiable at the extremum.

Claim

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

wlog Let this refer to min

Proof: Let $p < c$ & $q > c$ st $p, q \in I(c)$

$f(p) \geq f(c) \Rightarrow \frac{f(p) - f(c)}{p - c} \leq 0$

$f'(c) \leq 0 \Leftarrow$ ($p \to c$)

$f(q) \geq f(c) \Rightarrow \frac{f(q) - f(c)}{q - c} \geq 0$

$f'(c) \geq 0 \Leftarrow$ ($q \to c$)

[1] By virtue of the *squeeze* or *sandwich theorem*

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Claim

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

Proof: Suppose $f(c) \geq f(x)$ for all x in an open interval $\mathcal{I}$ containing c and that $f'(c)$ exists. Then the difference quotient $\frac{f(c+h)-f(c)}{h} \leq 0$ for small $h \geq 0$ (so that $c + h \in \mathcal{I}$). This inequality remains true as $h \to 0$ from the right. In the limit, $f'(c) \leq 0$.

[1] By virtue of the *squeeze* or *sandwich theorem*

First Derivative Test

First derivative test for local extreme value of f, when f is differentiable at the extremum.

Claim

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

Proof: Suppose $f(c) \geq f(x)$ for all x in an open interval $\mathcal{I}$ containing c and that $f'(c)$ exists. Then the difference quotient $\frac{f(c+h)-f(c)}{h} \leq 0$ for small $h \geq 0$ (so that $c + h \in \mathcal{I}$). This inequality remains true as $h \to 0$ from the right. In the limit, $f'(c) \leq 0$. Also, the difference quotient $\frac{f(c+h)-f(c)}{h} \geq 0$ for small $h \leq 0$ (so that $c + h \in \mathcal{I}$). This inequality remains true as $h \to 0$ from the left. In the limit, $f'(c) \geq 0$. Since $f'(c) \leq 0$ as well as $f'(c) \geq 0$, we must have $f'(c) = 0$[1]. □

[1] By virtue of the *squeeze* or *sandwich theorem*

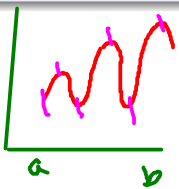
The Extreme Value Theorem

Sketch: Let $[a,b]$ = union of finite closed intervals, on each of which f is monotone

A most fundamental theorems in calculus concerning continuous functions on closed intervals.

Claim

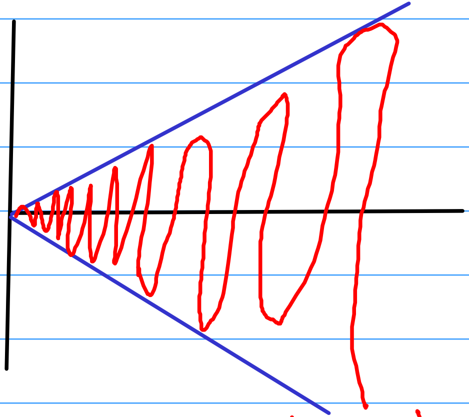
A continuous function $f(x)$ on a closed and bounded interval $[a, b]$ attains a minimum value $f(c)$ for some $c \in [a, b]$ and a maximum value $f(d)$ for some $d \in [a, b]$. That is, a continuous function on a closed, bounded interval attains a minimum and a maximum value.



$x\sin\left(\frac{1}{x}\right)$ on $[0,1]$

0 if $x=0$

$$f(x) = x \sin\left(\frac{1}{x}\right) \quad \text{if } x \in (0,1]$$
$$= 0 \quad \text{if } x = 0$$



This is where compactness etc help in $\mathbb{R}^n$

The Extreme Value Theorem (contd.)

We must point out that either or both of the values c and d may be attained at the end points of the interval $[a, b]$. Based on theorem (1), the extreme value theorem can extended as:

Claim

A continuous function $f(x)$ on a closed and bounded interval $[a, b]$ attains a minimum value $f(c)$ for some $c \in [a, b]$ and a maximum value $f(d)$ for some $d \in [a, b]$. If $a < c < b$ and $f'(c)$ exists, then $f'(c) = 0$. If $a < d < b$ and $f'(d)$ exists, then $f'(d) = 0$.

Proof sketch: In 4 parts. In $\Re^n$, one additionally needs compactness of the set in order to get this result.

Rolle's Theorem

① Extreme value thm (evt)

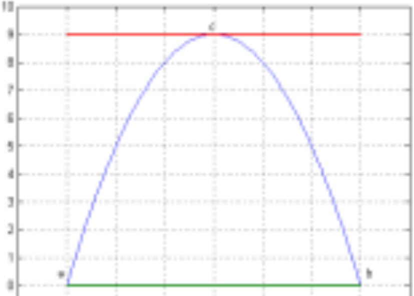
Claim

② If $f(x)=f(a)=f(b)=k \Rightarrow f'=0$

If f is continuous on $[a, b]$ and differentiable at all $x \in (a, b)$ and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

This result can be easily proved using the Extreme value theorem. Figure 1 illustrates Rolle's theorem with an example function $f(x) = 9 - x^2$ on the interval $[-3, +3]$.

③ If $f(x)$ is not const, by evt & critical pt test

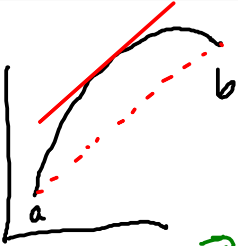


Mean Value Theorem

A generalization of the Rolle's theorem and proved using the Rolle's theorem:

Claim

If f is continuous on $[a, b]$ and differentiable at all $x \in (a, b)$, then there is some $c \in (a, b)$ such that, $f'(c) = \frac{f(b)-f(a)}{b-a}$.



$$g(x) = f(x) - \frac{f(b)-f(a)}{(b-a)}(x-a)$$

$$g(a) = g(b) = f(a)$$

$$\Rightarrow \exists c \quad g'(c) = 0$$

Mean Value Theorem

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Claim

If f is continuous on $[a, b]$ and differentiable at all $x \in (a, b)$, then there is some $c \in (a, b)$ such that, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof: Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$ on $[a, b]$. We note rightaway that $g(a) = g(b)$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Applying Rolle's theorem on $g(x)$, we know that there exists $c \in (a, b)$ such that $g'(c) = 0$. Which implies that $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

Mean Value Theorem (contd.)

Figure 2 illustrates the mean value theorem for $f(x) = 9 - x^2$ on the interval $[-3, 1]$. We observe that the tangent at $x = -1$ is parallel to the secant joining -3 to 1. That is, $f'(-1) = \frac{f(1) - f(-3)}{4}$ One could think of the *mean value theorem* as a slanted version of Rolle's theorem.

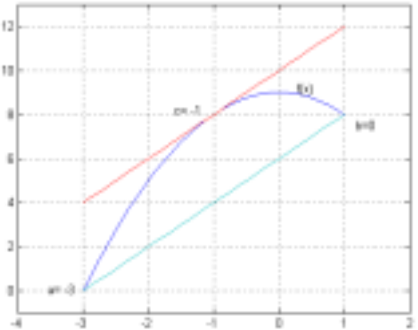


Figure 2

Corollary and Approximations

A natural corollary of Mean Value Theorem is as follows:

Corollary

Let f be continuous on $[a, b]$ and differentiable on (a, b) with $m \leq f'(x) \leq M$, $\forall x \in (a, b)$. Then,
$m(x - t) \leq f(x) - f(t) \leq M(x - t)$, if $a \leq t \leq x \leq b$.

Corollary and Approximations (contd.)

Let $\mathcal{D}$ be the domain of function f. We define

1. the linear approximation of a differentiable function $f(x)$ as $L_a(x) = f(a) + f'(a)(x - a)$ for some $a \in \mathcal{D}$. We note that $L_a(x)$ and its first derivative at a agree with $f(a)$ and $f'(a)$ respectively.
2. the quadratic approximatin of a twice differentiable function $f(x)$ as the parabola $Q_a(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$. We note that $Q_a(x)$ and its first and second derivatives at a agree with $f(a)$, $f'(a)$ and $f''(a)$ respectively.
3. the cubic approximation of a thrice differentiable function $f(x)$ is $C_a(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \frac{1}{6}f'''(a)(x - a)^3$. $C_a(x)$ and its first, second and third derivatives at a agree with $f(a)$, $f'(a)$, $f''(a)$ and $f'''(a)$ respectively.

Convexity and Concavity of Approximations

The parabola given by $Q_a(x)$ is strictly convex if $f''(a) > 0$ and is strictly concave if $f''(a) < 0$. The coefficient of x^2 in $Q_a(x)$ is $\frac{1}{2}f''(a)$. Figure 3 illustrates the linear, quadratic and cubic approximations to the function $f(x) = \frac{1}{x}$ with $a = 1$.

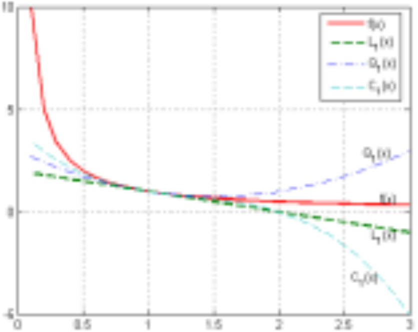


Figure 3:

Taylor's Theorem and n^{th} degree polynomial approximation

The n^{th} degree polynomial approximation of a function is used to prove a generalization of the mean value theorem, called the *Taylor's theorem*.

Claim

The Taylor's theorem states that if f and its first n derivatives $f', f'', \ldots, f^{(n)}$ *are continuous on the closed interval* $[a, b]$, *and differentiable on* (a, b), *then there exists a number* $c \in (a, b)$ *such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(b-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(b-a)^{n+1}$$

Proof:

Define

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

and

$$\phi_n(x) = p_n(x) + \Gamma(x-a)^{n+1}$$

The polynomials $p_n(x)$ as well as $\phi_n(x)$ and their first n derivatives match f and its first n derivatives at $x = a$. We will choose a value of Γ so that

$$f(b) = p_n(b) + \Gamma(b-a)^{n+1}$$

This requires that $\Gamma = \frac{f(b)-p_n(b)}{(b-a)^{n+1}}$.

Taylor's Theorem and n^{th} degree polynomial approximation

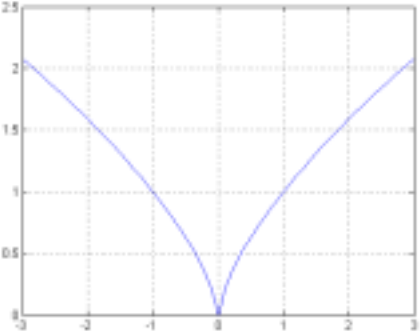
Define the function $g(x) = f(x) - \phi_n(x)$ that measures the difference between function f and the approximating function $\phi_n(x)$ for each $x \in [a, b]$.

- Since $g(a) = g(b) = 0$ and since g and g' are both continuous on $[a, b]$, we can apply the Rolle's theorem to conclude that there exists $c_1 \in [a, b]$ such that $g'(c_1) = 0$.
- Similarly, since $g'(a) = g'(c_1) = 0$, and since g' and g'' are continuous on $[a, c_1]$, we can apply the Rolle's theorem to conclude that there exists $c_2 \in [a, c_1]$ such that $g''(c_2) = 0$.
- In this way, Rolle's theorem can be applied successively to $g'', g''', \ldots, g^{(n+1)}$ to imply the existence of $c_i \in (a, c_{i-1})$ such that $g^{(i)}(c_i) = 0$ for $i = 3, 4, \ldots, n+1$. Note however that $g^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!\Gamma$ which gives us another representation 'of Γ as $\frac{f^{(n+1)}(c_{n+1})}{(n+1)!}$.

Thus

Mean Value, Taylor's Theorem and words of caution

Note that if f fails to be differentiable at even one number in the interval, then the conclusion of the mean value theorem may be false. For example, if $f(x) = x^{2/3}$, then $f'(x) = \frac{2}{3\sqrt[3]{x}}$ and the theorem does not hold in the interval $[-3, 3]$, since f is not differentiable at s0 as can be seen in Figure 4.



Sufficient Conditions for Increasing and decreasing functions

A function f is said to be ...

- *increasing* on an interval $\mathcal{I}$ in its domain $\mathcal{D}$ if $f(t) < f(x)$ whenever $t < x$.
- *decreasing* on an interval $\mathcal{I} \in \mathcal{D}$ if $f(t) > f(x)$ whenever $t < x$.

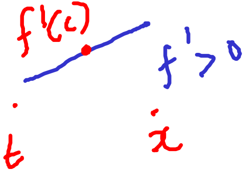
Consequently:

Claim

Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on $int(\mathcal{I})$. Then:

1. *if $f'(x) > 0$ for all $x \in int(\mathcal{I})$, then f is increasing on $\mathcal{I}$;*
2. *if $f'(x) < 0$ for all $x \in int(\mathcal{I})$, then f is decreasing on $\mathcal{I}$;*
3. *if $f'(x) = 0$ for all $x \in int(\mathcal{I})$, iff, f is constant on $\mathcal{I}$.*

Proof



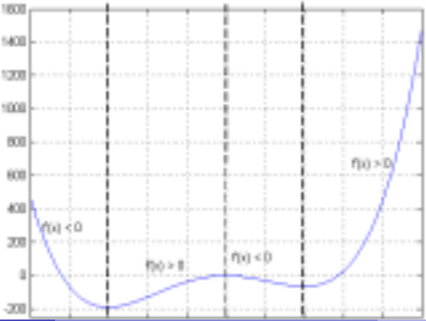
Proof:
Let $t \in \mathcal{I}$ and $x \in \mathcal{I}$ with $t < x$. By virtue of the mean value theorem, $\exists c \in (t, x)$ such that $f'(c) = \frac{f(x)-f(t)}{x-t}$.

- If $f'(x) > 0$ for all $x \in int(\mathcal{I})$, $f'(c) > 0$, which implies that $f(x) - f(t) > 0$ and we can conclude that f is increasing on $\mathcal{I}$.
- If $f'(x) < 0$ for all $x \in int(\mathcal{I})$, $f'(c) < 0$, which implies that $f(x) - f(t) < 0$ and we can conclude that f is decreasing on $\mathcal{I}$.
- If $f'(x) = 0$ for all $x \in int(\mathcal{I})$, $f'(c) = 0$, which implies that $f(x) - f(t) = 0$, and since x and t are arbitrary, we can conclude that f is constant on $\mathcal{I}$.

□

Illustration

Figure 5 illustrates the intervals in $(-\infty, \infty)$ on which the function $f(x) = 3x^4 + 4x^3 - 36x^2$ is decreasing and increasing. First we note that $f(x)$ is differentiable everywhere on $(-\infty, \infty)$ and compute $f'(x) = 12(x^3 + x^2 - 6x) = 12(x-2)(x+3)x$, which is negative in the intervals $(-\infty, -3]$ and $[0, 2]$ and positive in the intervals $[-3, 0]$ and $[2, \infty)$. We observe that f is decreasing in the intervals $(-\infty, -3]$ and $[0, 2]$ and while it is increasing in the intervals $[-3, 0]$ and $[2, \infty)$.



Another sufficient condition for increasing/decreasing function

A related sufficient condition for a function f to be increasing/decreasing on an interval $\mathcal{I}$:

Claim

Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on int$(\mathcal{I})$. Then:

1. *if $f'(x) \geq 0$ for all $x \in$ int$(\mathcal{I})$, and if $f'(x) = 0$ at only finitely many $x \in \mathcal{I}$, then f is increasing on $\mathcal{I}$;*
2. *if $f'(x) \leq 0$ for all $x \in$ int$(\mathcal{I})$, and if $f'(x) = 0$ at only finitely many $x \in \mathcal{I}$, then f is decreasing on $\mathcal{I}$.*

For example, the derivative of the function $f(x) = 6x^5 - 15x^4 + 10x^3$ vanishes at 0, and 1 and $f'(x) > 0$ elsewhere. So $f(x)$ is increasing on $(-\infty, \infty)$.

Necessary conditions for increasing/decreasing function

The conditions for increasing and decreasing properties of $f(x)$ in theorem 7 are not necesssary. Figure 6 shows that for the function $f(x) = x^5$, though $f(x)$ is increasing in $(-\infty, \infty)$, $f'(0) = 0$.

$f'(x) \geq 0$

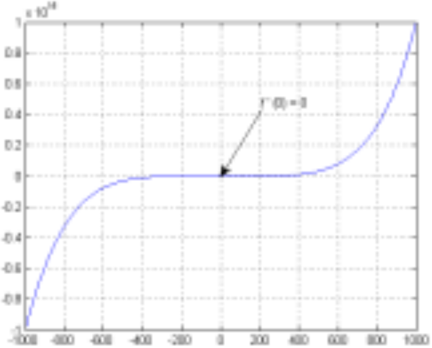


Figure 6:

Necessary conditions for increasing/decreasing function (contd.)

We have a slightly different necessary condition..

Claim

Let $\mathcal{I}$ be an interval, and suppose f is continuous on $\mathcal{I}$ and differentiable in int$(\mathcal{I})$. Then:

1. *if f is increasing on $\mathcal{I}$, then $f'(x) \geq 0$ for all $x \in$ int$(\mathcal{I})$;*
2. *if f is decreasing on $\mathcal{I}$, then $f'(x) \leq 0$ for all $x \in$ int$(\mathcal{I})$.*

Proof:

$$f(y) > f(x) \quad \forall\, y > x$$

$$\Rightarrow \frac{f(y) - f(x)}{y - x} > 0 \xrightarrow{\lim_{y \to x}} f'(x) \geq 0$$

Necessary conditions for increasing/decreasing function (contd.)

We have a slightly different necessary condition..

Claim

Let $\mathcal{I}$ be an interval, and suppose f is continuous on $\mathcal{I}$ and differentiable in $int(\mathcal{I})$. Then:

1. *if f is increasing on $\mathcal{I}$, then $f'(x) \geq 0$ for all $x \in int(\mathcal{I})$;*
2. *if f is decreasing on $\mathcal{I}$, then $f'(x) \leq 0$ for all $x \in int(\mathcal{I})$.*

Proof: Suppose f is increasing on $\mathcal{I}$, and let $x \in int(\mathcal{I})$. Then $\frac{f(x+h)-f(x)}{h} > 0$ for all h such that $x+h \in int(\mathcal{I})$. This implies that $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} \geq 0$. For the case when f is decreasing on $\mathcal{I}$, it can be similarly proved that $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h} \leq 0$. □

Critical Point

This concept will help us derive the general condition for local extrema.

Definition

[Critical Point]: *A point c in the domain $\mathcal{D}$ of f is called a critical point of f if either $f'(c) = 0$ or $f'(c)$ does not exist.*

The following general condition for local extrema extends the result in theorem 1 to general non-differentiable functions.

Claim

If $f(c)$ is a local extreme value, then c is a critical number of f.

The converse of theorem 10 does not hold (see Figure 6); 0 is a critical number ($f'(0) = 0$), although $f(0)$ is not a local extreme value.

Critical Point and Local Extreme Value

Given a critical point c, the following test helps determine if $f(c)$ is a local extreme value:

Procedure

[Local Extreme Value]: *Let c be an isolated critical point of f*

1. *$f(c)$ is a local minimum if $f(x)$ is decreasing in an interval $[c-\epsilon_1, c]$ and increasing in an interval $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ is increasing in an interval $[c-\epsilon_1, c]$ and decreasing in an interval $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*

Given a critical point c, *first derivative test* (sufficient condition) helps determine if $f(c)$ is a local extreme value:

Procedure

[First derivative test]: *Let c be an isolated critical point of f*

1. *$f(c)$ is a local minimum if the sign of $f'(x)$ changes from negative in $[c-\epsilon_1, c]$ to positive in $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ the sign of $f'(x)$ changes from positive in $[c-\epsilon_1, c]$ to negative in $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
3. *If $f'(x)$ is positive in an interval $[c-\epsilon_1, c]$ and also positive in an interval $[c, c-\epsilon_2]$, or $f'(x)$ is negative in an interval $[c-\epsilon_1, c]$ and also negative in an interval $[c, c-\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$, then $f(c)$ is not a local extremum.*