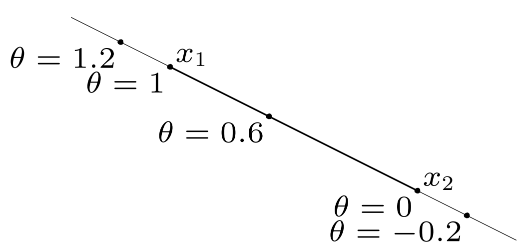
2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$ insight from linear algebra on geometry etc?

(conversely, every affine set can be expressed as solution set of system of linear equations)

For answer: pages 145 to 181 of

http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

underconstrained prob

Rank of A

over constrained problem

r=m=n	r=m<n	r=n<m
R=I	R=[I F]	$R=[I\ 0]^T$
Unique solution	Infinitely many solutions	0 or 1 solution

Figure 3.3: Summary of the properties of the solutions to the system of equations $Ax = b$.

$A_{m \times n}$ matrix

$$A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Example:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

(Gauss Elimination)

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \overset{E_{2,1},E3,1}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\overset{E3,2}{\Longrightarrow} \begin{bmatrix} [1] & 2 & 2 & 2 & b_1 \\ 0 & 0 & [2] & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

Condition for solvability: $b_3 - b_1 - b_2 = 0$

Thus:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable when $\mathbf{b}$ is in the column space $C(A)$.

Another way of describing solvability is:

The system of equations $A\mathbf{x} = \mathbf{b}$ is solvable if a combination of the rows of A produces a zero row, the requirement on $\mathbf{b}$ is that the same combination of the components of $\mathbf{b}$ has to yield zero.

Steps to find $x_{particular}$:

1. $\mathbf{x}_{particular}$[2]: Set all free variables (corresponding to columns with no pivots) to 0. In the example above, we should set $x_2 = 0$ and $x_4 = 0$.

2. Solve $A\mathbf{x} = \mathbf{b}$ for pivot variables.

In this example:

$$x_1 + 2x_3 = b_1 \quad \& \quad 2x_3 = b_2 - 2b_1$$

$$\Rightarrow \quad \mathbf{x}_{particular} = \begin{bmatrix} b_2 + 3b_1 \\ 0 \\ \frac{b_2 - 2b_1}{2} \\ 0 \end{bmatrix}$$

Now: $\mathbf{x}_{complete} = \mathbf{x}_{particular} + \mathbf{x}_{nullspace}$ since

$x_{nullspace}$ is s.t $Ax_{nullspace} = 0$

$$\boxed{A\mathbf{x}_{complete} = A(\mathbf{x}_{particular} + \mathbf{x}_{nullspace}) = \mathbf{b} + \mathbf{0} = \mathbf{b}}$$

Eg: if we choose $\mathbf{b} = [5\ 1\ 6]^T$, we get $\mathbf{x}_{particular} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$ &

$$x_{complete} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}}_{x_{particular}} + \underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{x_{nullspace}} \quad (3.36)$$

(obtained from Gauss elim by setting $b=0$)

Show that $x_{complete} = \theta x_1 + (1-\theta) x_2$ for some $x_1, x_2 \in \mathbb{R}^4$ & $\theta \in \mathbb{R}$

We knew $\{x \mid Ax=b\}$ is an affine set

Q: What is a more generalised definition of affine sets?

Ans (Clue from linear algebra discussion on affine sets):

Affine set $A = a + L$ & $\dim(A) = \dim(L)$

Roughly, an affine set is a shifted vector space

$a \in A$

Like $x_{particular}$

L is a subspace (eg null space)

https://en.wikipedia.org/wiki/Vector_space

closed under addition & scalar multiplication

Like $x_{nullspace}$

Please understand / brush up following concepts from linear algebra:

(a) Vector space: https://en.wikipedia.org/wiki/Vector_space & sec 3.5 of http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

(b) Null space: Sec 3.5.2 from https://en.wikipedia.org/wiki/Vector_space

$\{x \mid Ax = 0\}$ is nullspace of A

(c) Independence, basis & dimension: Sec 3.7 from http://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

H/w: Present some basis for $\mathbb{R}^4$

↳ Q1: How do you verify that you have indeed generated a basis?

Ans: (a) Look at RREF after Gauss elimination & ensure 4 pivots

(b) Look at $\det([v_1\ v_2\ v_3\ v_4])$ and ensure it is $\neq 0$

(c) Ensure that basis has size $= 4$

More appropriate name when x_1 & x_2 are pts in real, finite dimensional Euclidian vector space $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$

Convex set

line segment between x_1 and x_2: all points

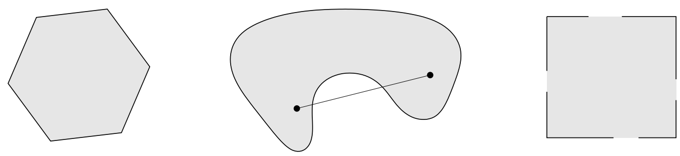
$$x = \theta x_1 + (1-\theta)x_2$$

with $0 \le \theta \le 1$ → For an affine set no constraints exist on θ. Hence affine set is convex (not vice versa)

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



Aside: Convex set is connected: https://en.wikipedia.org/wiki/Connected_space

Convex set can, but not necessarily contains '0'.

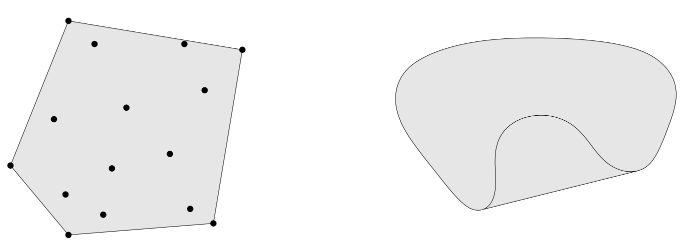
Convex combination and convex hull

convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \ge 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



Proof that any affine set is convex

- Let C be an affine set
- $\therefore$ Given any $x_1, x_2 \in C$, $\forall\, \theta \in \mathbb{R}$

$$\theta x_1 + (1-\theta) x_2 \in C$$

- $\Rightarrow$ Given any $x_1, x_2 \in C$, $\forall\, \theta \in [0,1]$

$$\theta x_1 + (1-\theta) x_2 \in C$$

$\Rightarrow$ C is convex

Converse is NOT true

$\therefore$ Every affine set is convex but every convex set is NOT affine

Computing the convex hull :

1) Requires a non-ambiguous representation of the required convex shape

Let n = # of input pts & h = # pts on convex hull.

ⓐ Simplest algo (Gift wrapping or Jarvis March). $O(nh)$

Guess: http://en.wikipedia.org/wiki/Gift_wrapping_algorithm

① Find 'leftmost' point P_1

② Pick next pt P_2 st all other pts lie to only one side of line segment P_1P_2

③ & so on . . . .

complexity = $O(nh)$

Try out applet: http://www.cs.princeton.edu/courses/archive/spr09/cos226/demo/ah/JarvisMarch.html

ⓑ Faster algo (Merge hull/Divide and conquer) $O(n \log n)$

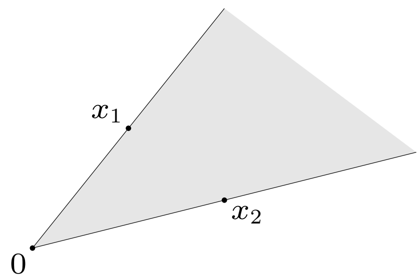
Try applet at http://students.cec.wustl.edu/~tpt1/cse546/HullApplet.htm

Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$



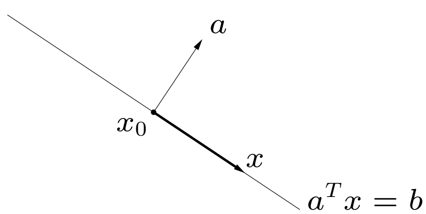
convex cone: set that contains all conic combinations of points in the set

Q: Relation between convex cone, convex set & affine set?

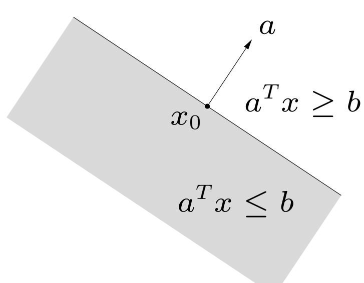
Ans: On next page.

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

In general for $x_1, x_2 \ldots x_n$

(a) Affine combination

$$\sum_{i=1}^{n} \theta_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \theta_i = 1$$

(b) Convex combination

$$\sum_{i=1}^{n} \theta_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} \theta_i = 1 \text{ \& } \theta_i \in [0,1]$$

(c) Conic combination

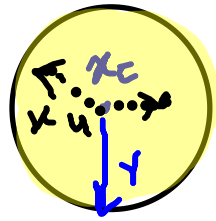
$$\sum_{i=1}^{n} \theta_i x_i \quad \text{s.t.} \quad \theta_i \geq 0$$

Set is called a, b or c if corresponding combination of pts in the set belongs to the set for all choices of θ under corresponding constraint

Easy to see that EVERY CONIC SET is CONVEX

& EVERY AFFINE SET is CONVEX

Euclidean balls and ellipsoids



(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

Eucledian or L_2 norm

ellipsoid: set of the form

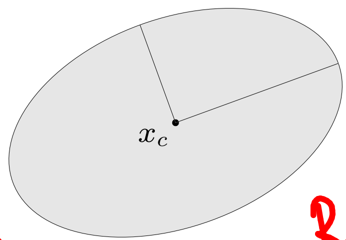
$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$$

u is a vector $\|u\| \le 1$

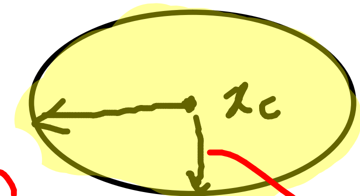
with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

achieves Rotation and rescaling

Diagonal P=> no rotation



Rotation



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Rescaling

H/W: Read up positive semidefiniteness

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

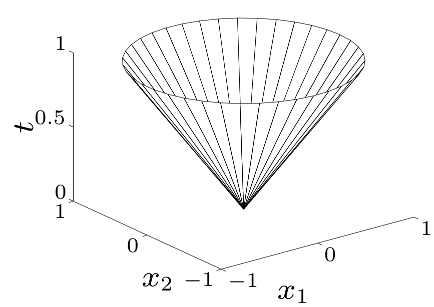
- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\mathrm{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

Q1: Is a norm cone a convex cone?

Ans:

Q2: Is every convex cone a norm cone?

Q3: Is there a convex cone that is also affine?