① LP is only a special case of CP with $K = \boxed{\mathbb{R}^+_n}$

② When will solution to CP = solution to CD?

CP: $\min_{x \in \mathbb{R}^n} \langle c, x \rangle$

s.t $Ax \geq_K b$

CD: $\max_{\lambda \in K^*} \langle b, \lambda \rangle$

s.t $A^* \lambda = c$

Since $K^{**} = K$, we saw that dual of CD is CP.

While there exist multiple ways of writing CP & CD, hereafter we pick another standard format (to help you get used to various representations)

CP: $\min \langle c, x \rangle_V$

s.t $Ax = b$

$x \in K \subseteq V$

F_p $A: V \to \mathbb{R}^n$

CD: $\max \langle b, \lambda \rangle_{\mathbb{R}^n}$

s.t $c - A^* \lambda \in K^*$

$\lambda \in \mathbb{R}^n$

F_d $K^* \subseteq V$

STRONG DUALITY THM:

CP infeasible
CD feasible & int.

① Let CP or CD be infeasible & $CP \to -\infty$ let other be feasible & have an interior. Then the other is unbounded

CD is infeasible
CP is feasible & int
$CD \to \infty$

② Let CP and CD be both feasible, and let one of them have an interior. Then there is 0 duality gap

③ Let CP and CD be both feasible and have interior. Then both have optimal solutions with 0 duality gap

2 special cases of strong conic duality

(Motivating Farkas' lemma)

① Instance of CP, $c=0$

$\{x \mid \underbrace{Ax=b,\ x\in K}_{CP^*}\}$ $\{\lambda \mid \langle b,\lambda\rangle>0,\ \underbrace{-A^*\lambda\in K^*,\ \lambda\in\mathbb{R}^n}_{CD^*}\}$

One of them is non-empty iff the other is empty: Theorem of alternatives or Farkas lemma - - - -(Version I)

② Instance of CD, $b=0$

$\{x \mid \underbrace{Ax=0,\ x\in K,\ \langle c,x\rangle<0}_{CP^*}\}$ $\{\lambda \mid \underbrace{c-A^*\lambda\in K^*,\ \lambda\in\mathbb{R}^n}_{CD^*}\}$.

Proof: (with blanks) VERSION I

We need the theorem of alternatives to prove strong conic duality [Also called Farkas' Lemma for Convex Cone]

Theorem of alternatives:

Consider $\{x \mid Ax = b,\ x \in K\}$ for a proper cone $K \subseteq V$ & $A : V \to \mathbb{R}^n$

Suppose $\exists\ \lambda$ s.t $-A^*\lambda \in \operatorname{int}(K^*)$. Then

$$\boxed{\begin{array}{ll} (a) \leftarrow & \{x \mid Ax = b\ x \in K\} \text{ has a feasible soln } x \quad \text{iff} \\ (b) \leftarrow & \{\lambda \mid -A^*\lambda \in K^*, \langle b, \lambda \rangle > 0\} \text{ has no feasible solution} \end{array}}$$

PROOF:

(i) $C = \{y = Ax \in \mathbb{R}^m, x \in K\}$ is a closed convex set

(ii) Let $\bar{\lambda}$ be s.t $-A^*\bar{\lambda} \in K^*$ and Let $\{x \mid Ax = b, x \in K\}$ have a feasible solution $\bar{x}$

$\Rightarrow\ -\langle \bar{\lambda}, b \rangle = -\langle \bar{\lambda}, A\bar{x} \rangle$ . . . .

. . . . ie $\{\lambda \mid -A^*\lambda \in K^*, \langle b, \lambda \rangle > 0\}$ has no solution

(iii) Let $\{x \mid Ax = b, x \in K\}$ have no feasible solution ie b ____ C

We will show that $\{\lambda \mid -A^*\lambda \in K^*, \langle \lambda, b \rangle > 0\}$ must be non-empty

Since C is a closed convex set, from the strict separating hyperplane theorem, $\exists\, \lambda \in \mathbb{R}^m$ s.t

$$> \langle \lambda, y \rangle \quad \forall\, y \in C$$

Since $\exists\, x \in K$ s.t $Ax = y$ for any $y \in C$

$$> \langle \lambda, Ax \rangle = \qquad \forall\, x \in K$$

- Thus, $\langle A^*\lambda, x \rangle$ is bounded above $\forall\, x \in K$
- Since $0 \in K$, $\langle b, \lambda \rangle > 0$
- Additionally, it must be that $\langle A^*\lambda, x \rangle \leq 0$ $\forall\, x \in K$.

  Otherwise if $\exists\, x \in K$ s.t $\langle A^*\lambda, x \rangle > 0$ then if $\alpha \to +\infty$ then $\to \infty$ contradicting that is bounded above for all x
- Since $\langle A^*\lambda, x \rangle \leq 0$ $\forall\, x$

$$\geq 0 \quad \forall\, x \Rightarrow \quad \in K^*.$$

- Thus, λ is a (feasible) solution for . - - - .

- .. complete the proof - - - -

Proof: (with blanks filled up) VERSION I

We need the theorem of alternatives to prove strong conic duality [Also called Farkas' Lemma for Convex Cone]

Theorem of alternatives:

Consider $\{x \mid Ax = b,\ x \in K\}$ for a proper cone $K \subseteq V$ & $A: V \to \mathbb{R}^n$

Suppose $\exists\ \lambda$ s.t $-A^*\lambda \in \text{int}(K^*)$. Then

$$\boxed{\begin{array}{l} \text{(a)} \leftarrow \{x \mid Ax = b\ x \in K\} \text{ has a feasible soln } x \quad \text{iff} \\ \text{(b)} \leftarrow \{\lambda \mid -A^*\lambda \in K^*, \langle b, \lambda \rangle > 0\} \text{ has no feasible solution} \end{array}}$$

PROOF:

(i) $C = \{y = Ax \in \mathbb{R}^m, x \in K\}$ is a closed convex set

(ii) Let $\bar{\lambda}$ be s.t $-A^*\bar{\lambda} \in K^*$ and Let $\{x \mid Ax = b, x \in K\}$ have a feasible solution $\bar{x}$

$$\Rightarrow -\langle \bar{\lambda}, b \rangle = -\langle \bar{\lambda}, A\bar{x} \rangle = \langle -A^*\bar{\lambda}, \bar{x} \rangle \geq 0$$

since $-A^*\bar{\lambda} \in K^*$ ... ie $\{\lambda \mid -A^*\lambda \in K, \langle b, \lambda \rangle > 0\}$ has no solution

(iii) Let $\{x \mid Ax = b, x \in K\}$ have no feasible solution ie $b \notin C$

We will show that $\{\lambda \mid -A^*\lambda \in K^*, \langle \lambda, b\rangle > 0\}$ must be non-empty

Since C is a closed convex set, from the strict separating hyperplane theorem, $\exists\, \lambda \in \mathbb{R}^m$ s.t

$$\langle \lambda, b\rangle > \langle \lambda, y\rangle \quad \forall\, y \in C$$

Since $\exists\, x \in K$ s.t $Ax = y$ for any $y \in C$

$$\langle \lambda, b\rangle > \langle \lambda, Ax\rangle = \langle A^*\lambda, x\rangle \quad \forall\, x \in K$$

- Thus, $\langle A^*\lambda, x\rangle$ is bounded above $\forall\, x \in K$
- Since $0 \in K$, $\langle \lambda, b\rangle > 0$
- Additionally, it must be that $\langle A^*\lambda, x\rangle \leq 0$ $\forall\, x \in K$.

  Otherwise if $\exists\, x \in K$ s.t $\langle A^*\lambda, x\rangle > 0$ then if $\alpha \to +\infty$ then $\langle A^*\lambda, \alpha x\rangle \to \infty$ contradicting that $\langle A^*\lambda, x\rangle$ is bounded above for all x
- Since $\langle A^*\lambda, x\rangle \leq 0$ $\forall\, x$

$$\langle -A^*\lambda, x\rangle \geq 0 \quad \forall\, x \Rightarrow -A^*\lambda \in K^*.$$

- Thus, λ is a (feasible) solution for $\{\lambda \mid -A^*\lambda \in K^*, \langle \lambda, b\rangle > 0\}$ which is thus non-empty

Proof: (With Blanks) VERSION II

Corollary of the Theorem of alternatives [Also called Farkas' Lemma for Convex Cone]

Consider

$\{(y,s) \mid \underbrace{c - A^{+}\lambda}_{c \in V} = s \in K\}$ for a proper cone $K \subseteq V$ & $A: V \to \mathbb{R}^n$ ($A^{+}: \mathbb{R}^n \to V$)

Suppose $\exists$ x s.t $Ax = 0$ $x \in \text{int}(K^*)$. Then

$$\boxed{\begin{array}{l} (a)\ \{(\lambda, s) \mid c - A^*\lambda = s \in K\} \text{ has a solution } (\lambda, s) \quad \text{iff} \\ (b)\ \{x \mid Ax = 0, x \in K^+, \langle c, x\rangle < 0\} \text{ has no feasible solution} \end{array}}$$

PROOF:

(i) $C = \{t = s + A^{+}\lambda, \lambda \in \mathbb{R}^n, s \in K\}$ is a closed convex set

(ii) Let $\bar{x} \in K^*$ be s.t $Ax = 0$ an let $\{(\lambda, s) \mid c - A^*\lambda = s \in K\}$ have a feasible solution $(\bar{\lambda}, \bar{s})$

$\Rightarrow \langle c - A^*\lambda, \bar{x}\rangle =$

i.e $\{x \mid Ax = 0, x \in K, \langle c, x\rangle < 0\}$ has no feasible solution

(iii) Let $\{(\lambda, s) \mid c - A^*\lambda = s \in K\}$ have no feasible solution, i.e

We will show that $\{x \,/\, Ax = 0, x \in K^*, \langle c, x\rangle < 0\}$ must be non-empty. Since C' is a closed convex set & $c \notin C'$ by strict separating hyperplane theorem, there exists $x \in V$ s.t

$$[redacted] < \langle x, t\rangle \quad \forall\, t \in C'$$

Since $\exists\, \lambda \in \mathbb{R}^n$ s.t $t = s + A^*\lambda \;\; \forall\, t \in C'$, we will have

$$[redacted] < \langle x, s + A^*\lambda\rangle = \langle x, s\rangle + \langle x, A^*\lambda\rangle = [redacted]$$

- Thus $\langle x, s + A^*\lambda\rangle$ is bounded above $\forall\, \lambda \in \mathbb{R}^n$ $\forall\, \lambda \in \mathbb{R}^n$
- Since $0 \in s + A^*\lambda$ for $s = 0 \in K$ & $\lambda = 0 \in \mathbb{R}^n$, [redacted]
- Additionally, it must be that $\langle x, s + A^*\lambda\rangle \geq 0 \;\; \forall\, \lambda \in \mathbb{R}^n$

  Otherwise if $\exists\, \lambda \in \mathbb{R}^n$ s.t $\langle x, s + A^*\lambda\rangle < 0$ then if $\alpha \to +\infty$, then [redacted] $\to -\infty$ contradicting that [redacted] is bounded below $\forall\, \lambda \in \mathbb{R}^n$
- Since $\langle x, s + A^*\lambda\rangle = \langle x, s\rangle + \langle Ax, \lambda\rangle \geq 0 \;\; \forall\, \lambda \in \mathbb{R}^n$

  $\hookrightarrow$ (1). since o/w

  $\langle x, s\rangle + \langle Ax, \beta\lambda\rangle \to -\infty$ for $\beta \to \infty$ or $\beta \to -\infty$

  (2). [redacted] $\Rightarrow$ [redacted]

  (3) [redacted]

  [redacted] $\Rightarrow$ [redacted]
- Thus, x is a (feasible) solution for

  $\{x \,/\, Ax = 0, x \in K^*, \langle c, x\rangle < 0\}$ which is therefore non-empty

Proof: (with blanks filled up) VERSION II

Also a Corollary of the Theorem of alternatives [Also called Farkas' Lemma for Convex Cone]

Consider

$\{(y,s) \mid \underbrace{c - A^* \lambda}_{c \in V} = s \in K\}$ for a proper cone $K \subseteq V$ & $A: V \to \mathbb{R}^n$ $(A^*: \mathbb{R}^n \to V)$

Suppose $\exists$ x s.t $Ax = 0$ $x \in \text{int}(K^*)$. Then

$$\boxed{\begin{array}{l} \text{(a) } \{(\lambda,s) \mid c - A^*\lambda = s \in K\} \text{ has a solution } (\lambda,s) \quad \text{iff} \\ \text{(b) } \{x \mid Ax=0, x \in K^*, \langle c,x\rangle < 0\} \text{ has no feasible solution} \end{array}}$$

PROOF:

(i) $C' = \{t = s + A^*\lambda, \lambda \in \mathbb{R}^n, s \in K\}$ is a closed convex set

(ii) Let $\bar{x} \in K^*$ be s.t $Ax = 0$ an let $\{(\lambda,s) \mid c - A^*\lambda = s \in K\}$ have a feasible solution $(\bar{\lambda}, \bar{s})$

$\Rightarrow \langle c - A^*\lambda, \bar{x}\rangle = \langle c, \bar{x}\rangle - \langle A^*\lambda, \bar{x}\rangle = \langle c, \bar{x}\rangle - \langle \bar{y}, A^*\lambda\rangle = \langle c, \bar{x}\rangle \geq 0$

i.e $\{x \mid Ax = 0, x \in K, \langle c, x\rangle < 0\}$ has no feasible solution

(iii) Let $\{(\lambda,s) \mid c - A^*\lambda = s \in K\}$ have no feasible solution, i.e $c \notin C'$

We will show that $\{x / Ax = 0, x \in K^*, \langle c, x\rangle < 0\}$ must be non-empty. Since C' is a closed convex set & $c \notin C'$ by strict separating hyperplane theorem, there exists $x \in V$ s.t $\langle x, c\rangle < \langle x, t\rangle \ \forall\, t \in C'$

Since $\exists \lambda \in \mathbb{R}^n$ s.t $t = s + A^*\lambda \ \forall\, t \in C'$, we will have

$$\langle x, c\rangle < \langle x, s + A^*\lambda\rangle = \langle x, s\rangle + \langle x, A^*\lambda\rangle = \langle x, s\rangle + \langle Ax, \lambda\rangle \quad \forall\, \lambda \in \mathbb{R}^n$$

- Thus $\langle x, s + A^*\lambda\rangle$ is bounded above $\forall\, \lambda \in \mathbb{R}^n$
- Since $0 \in s + A^*\lambda$ for $s = 0 \in K$ & $\lambda = 0 \in \mathbb{R}^n$, $\langle x, c\rangle < 0$
- Additionally, it must be that $\langle x, s + A^*\lambda\rangle \geq 0 \ \forall\, \lambda \in \mathbb{R}^n$
  Otherwise if $\exists\, \lambda \in \mathbb{R}^n$ s.t $\langle x, s + A^*\lambda\rangle < 0$ then if $\alpha \to +\infty$, then $\langle x, \alpha(s + A^*\lambda)\rangle \to -\infty$ contradicting that $\langle x, s + A^*\lambda\rangle$ is bounded below $\forall\, \lambda \in \mathbb{R}^n$
- Since $\langle x, s + A^*\lambda\rangle = \langle x, s\rangle + \langle Ax, \lambda\rangle \geq 0 \ \forall\, \lambda \in \mathbb{R}^n$
  - ↳ ① $\langle Ax, \lambda\rangle = 0 \ \forall\, \lambda$ since o/w $\langle x, s\rangle + \langle Ax, \beta\lambda\rangle \to -\infty$ for $\beta \to \infty$ or $\beta \to -\infty$
  - ② $\langle Ax, \lambda\rangle = 0 \ \forall\, \lambda \Rightarrow Ax = 0$
  - ③ If $\langle Ax, \lambda\rangle = 0$ & $\langle x, s\rangle + \langle Ax, \lambda\rangle \geq 0 \ \forall\, \lambda \in \mathbb{R}^n$ then $\langle x, s\rangle \geq 0 \Rightarrow x \in K^*$ ($\because s \in K$)
- Thus, x is a (feasible) solution for $\{x / Ax = 0, x \in K^*, \langle c, x\rangle < 0\}$ which is therefore non-empty

Now we apply theorem of alternatives (Farkas' lemma for conic inequalities) to prove the strong conic duality theorem

① If $\mathcal{F}_d$ is empty and $\mathcal{F}_p$ is feasible & has an interior feasible solution, then we have $(\hat{x} \in \mathrm{int}(K)$ & $\hat{\tau} = 1)$ form an interior

feasible solution to

$$Ax - b\tau = 0 \quad (\hat{x}, \hat{\tau}) \in \mathrm{int}(K \oplus \mathbb{R}_{++})$$

we can now form an alternative system pair based on Farkas' lemma II

$$① \left\{ (x,\tau) \,\middle|\, Ax - b\tau = 0,\ \langle (c,-z),(x,\tau) \rangle < 0,\ (x,\tau) \in K \oplus \mathbb{R}_+ \right\}$$

$$② \left\{ (\lambda,\theta) \,\middle|\, \underbrace{c - A^*\lambda \in K^*,\ -\langle b,\lambda \rangle}\ + \theta = -z,\ \theta \in \mathbb{R}_+ \right\}$$

if $s = c - A^*\lambda$ then $(s,\theta) \in K^* \oplus \mathbb{R}_+$

② is infeasible $\Rightarrow$ ① must have a feasible soln (x,τ). If at (x,τ) $\tau > 0$ then $\frac{1}{\tau} Ax = b$ & $x/\tau \in K$ & $\langle c, x/\tau \rangle < z$ for any z

$\Rightarrow$

Otherwise, $\tau = 0 \Rightarrow$ a new soln $\hat{x} + \alpha x$ is

$\Rightarrow$ Objective of CP is unbounded from below

(2) Let $\mathcal{F}_p$ be feasible & have an interior feasible

(A) soln & z be its infimum.

- We will make use of alternate system pair (*) from part (1) (since infimum is (z)).
- Now is infeasible $\Rightarrow$

From weak duality theorem

$\Rightarrow$

i.e we have soln (λ, s) s.t

$$A^* \lambda + s = c \qquad \langle b, \lambda \rangle = z \qquad s \in K^*$$

(B) SIMILARLY, one can prove that if z = supremum of dual (since CP & CD are both assumed to be feasible) and if F_D has non-empty interior then $\exists x \in K$ s.t $\langle c, x \rangle = z$

(3) If F_p & F_D are both feasible & both have interior, we can apply both parts (A) and (B) of above claim to get $\bar{x} \in F_p$ & $\bar{\lambda} \in F_D$ s.t $\inf CP = \langle \bar{x}, c \rangle = \langle \bar{\lambda}, b \rangle = \sup CD$
$\Rightarrow$ CP & CD have attainable optimal solns