$$\begin{cases} \min f(x) \\ \text{s.t. } g_i(x) \le 0 \quad i = 1 \dots m \\ h_j(x) = 0 \quad j = 1 \dots k \end{cases}$$

We will generalize the inequalities & equalities

$$\min_x f(x) \quad \text{s.t. } g_i(x) \le 0, \; h_j(x) = 0$$

$$\geq \min_x \max_{\lambda, \mu} \underbrace{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{k} \mu_j h_j(x)}_{L(x,\lambda,\mu)} \quad \text{s.t. } g_i(x) \le 0, \; h_j(x) = 0, \; \lambda_i \ge 0, \; \mu_j \in \mathbb{R}$$

$$\geq \min_x \max_{\lambda,\mu} L(x,\lambda,\mu) \quad \lambda_i \ge 0, \; \mu_j \in \mathbb{R}$$

under strong duality

$\lambda_i^* g_i(x^*) = 0 \quad \forall i$

$\lambda_i^* \ge 0$

$\mu_j^* h_j(x^*) = 0 \quad \forall j$

$$\geq \max_{\substack{\lambda,\mu \\ \lambda_i \ge 0 \\ \mu_j \in \mathbb{R}}} \underbrace{\min_x L(x,\lambda,\mu)}_{L^*(\lambda,\mu) \text{ or lagrange dual fn.}}$$

$$= \underbrace{\max_{\lambda,\mu,\lambda\ge 0} L^*(\lambda,\mu)}_{\text{Dual opt problem}}$$

General weak duality result

$$\min_{x \in \mathcal{D}} f(x)$$

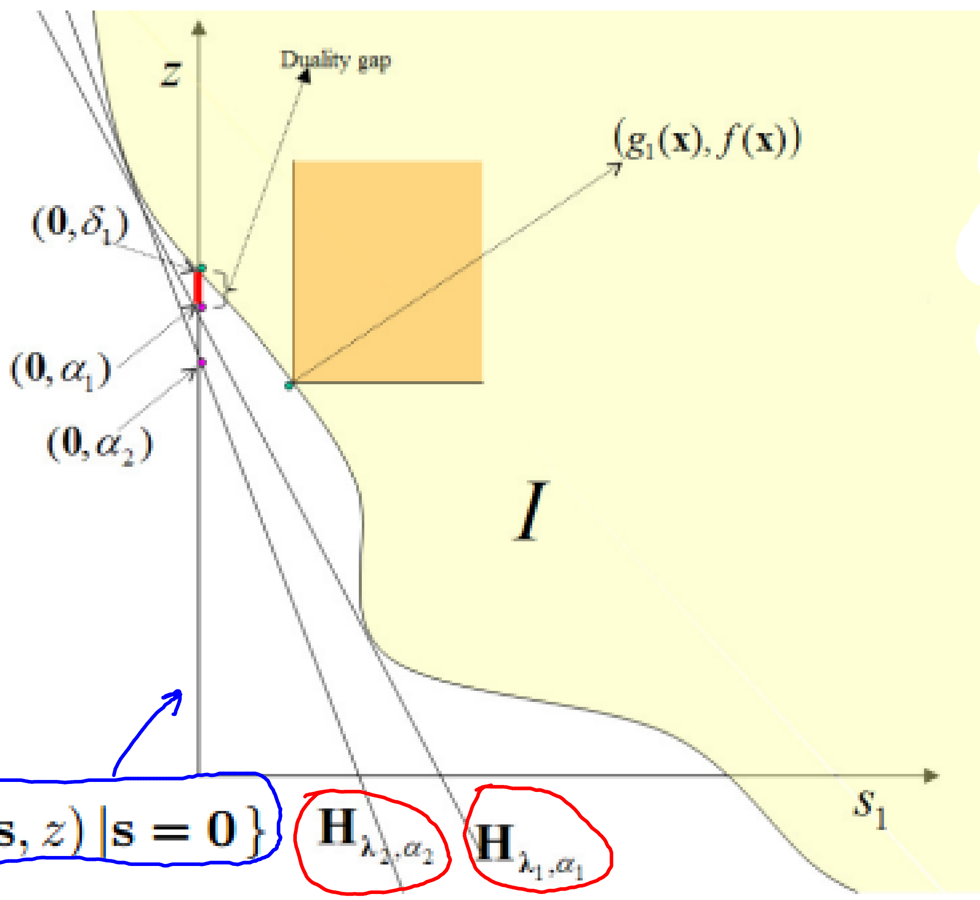
$$\text{s.t } g_i(x) \le 0 \quad i = 1 \ldots m$$

(The general dual problem & its geometric interpretation)

pg 292, sec 4.4.3 of http://www.cse.iitb.ac.in/~cs709/notes/BasicsOfConvexOptimization.pdf

Consider the set:

$$\mathcal{I} = \{(\mathbf{s}, z) \mid \mathbf{s} \in \Re^m,\ z \in \Re,\ \exists \mathbf{x} \in \mathcal{D} \text{ with } g_i(\mathbf{x}) \le s_i\ \forall 1 \le i \le m,\ f(\mathbf{x}) \le z\}$$



$$\mathcal{L} = \{(\mathbf{s}, z) \mid \mathbf{s} = \mathbf{0}\}$$

$$\mathcal{H}_{\lambda,\alpha} = \{(\mathbf{s}, z) \mid \lambda^T.\mathbf{s} + z = \alpha\}$$

Smallest z value in $\mathcal{I}$ for $s_1 \le 0$ will be for $s_1 = 0$

Since $(s_1, z_1) \in \mathcal{I}$
$\Rightarrow (s_2, z_2) \in \mathcal{I}$
$\forall s_2 \ge s_1 \ \& \ z_2 \ge z_1$

Thus $\Rightarrow$ is not possible!

$$\begin{array}{ll} \max & \alpha \\ \text{subject to} & \mathcal{H}^+_{\lambda,\alpha} \supseteq \mathcal{I} \end{array}$$

$$\begin{array}{ll} \max & \alpha \\ \text{subject to} & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I} \end{array}$$

a

$$\begin{array}{ll} \max & \alpha \\ \text{subject to} & \lambda^T.\mathbf{s} + z \geq \alpha \ \forall (\mathbf{s}, z) \in \mathcal{I} \\ & \lambda \geq \mathbf{0} \end{array}$$

If $\exists\ \alpha_b$ soln to b,
s.t. $\lambda^T g(x_b) + f(x_b) = \alpha_b$
then $s = g(x_b)$ $z = f(x_b)$
is soln to a
complete proof of equivalence

b

$$\begin{array}{ll} \max & \alpha \\ \text{subject to} & \lambda^T.\mathbf{g}(\mathbf{x}) + f(\mathbf{x}) \geq \alpha \ \forall \mathbf{x} \in \mathcal{D} \\ & \lambda \geq \mathbf{0} \end{array}$$

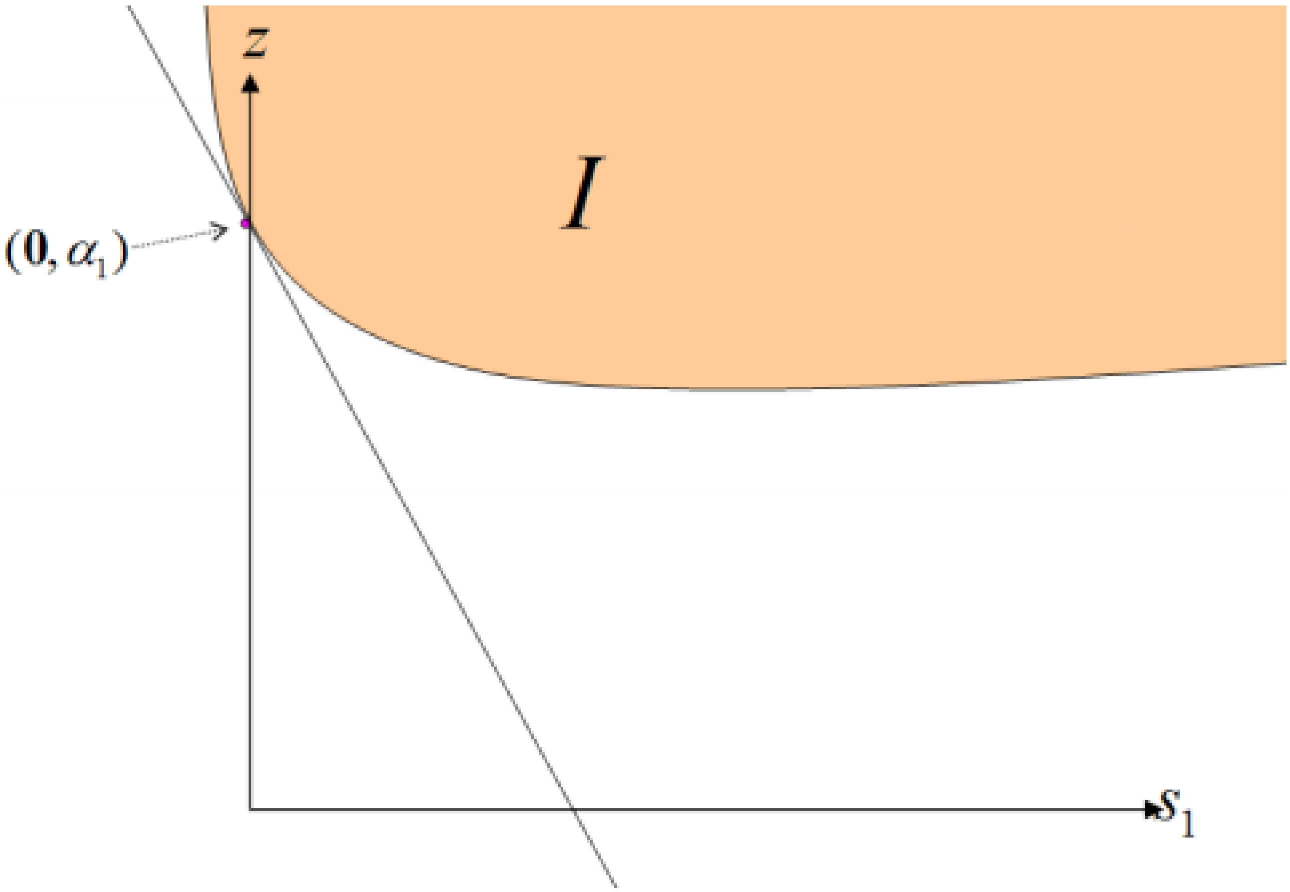
$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & L(\mathbf{x}, \lambda) \geq \alpha \ \forall \mathbf{x} \in \mathcal{D} \\
 & \lambda \geq \mathbf{0}
\end{array}
$$

Since, $L^*(\lambda) = \min_{\mathbf{x} \in \mathcal{D}} L(\mathbf{x}, \lambda)$, we can deal with the equivalent

$$
\begin{array}{ll}
\max & \alpha \\
\text{subject to} & L^*(\lambda) \geq \alpha \\
 & \lambda \geq \mathbf{0}
\end{array}
$$

This problem can be restated as

$$
\begin{array}{ll}
\max & L^*(\lambda) \\
\text{subject to} & \lambda \geq \mathbf{0}
\end{array}
$$



Q: What is desirable of the set I for zero duality gap?

Ans: $\exists\ (0,\alpha) \in I$ and λ s.t $\lambda^T s + z \geq \alpha \ \forall\ (s,z) \in I$

& Intersection of I with z axis is closed below

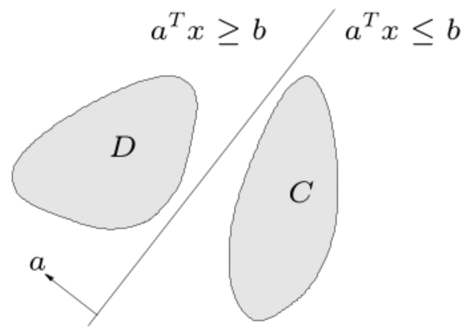
$\Leftrightarrow$ $\exists$ a supporting hyperplane to I at $(0,\alpha)$
& Intersection of I with z axis is closed below (with $(0,\alpha)$ being boundary pt)

$\Leftarrow$ I is closed & $\exists$ a supporting hyperplane to I at every boundary point

Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

consequence

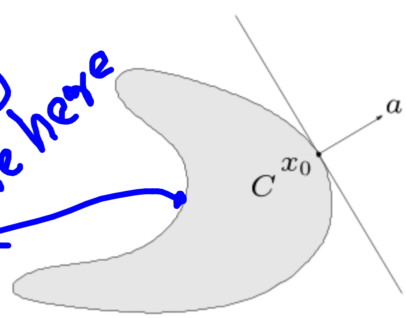
Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

no supporting hyperplane here



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Proof (from separating hyperplane theorem):

ⓐ interior$(C) \neq \phi$ (so that $\text{int}(C) \cap \{x_0\} = \phi$)

Apply separating hyperplane theorem to sets $C' = \{x_0\}$ and $D' = \text{interior}(C)$

Strict separation not applicable since int(C) is open

$\exists\, a, b$ s.t. $a^T x \geq b \;\forall x \in D'$

$a^T x_0 \leq b$

This inequality extends to all boundary (limit pts) leading to $a^T x_0 \geq b$

combined: $a^T x_0 = b$

ⓑ $\text{int}(C) = \phi$

$\Rightarrow$ Any hyperplane containing that affine set contains C & x_0 (H/w: Prove)

$\Rightarrow$ This hyperplane is a trivial supporting hyperplane

Some topological concepts:

[Topological set is set with concept of nbrhood]

① A set U is called an open set if it does not contain any of its boundary pts. If S is a metric space (eg an inner product space) with distance metric $d(x,y)$, then a subset U of S is called open if, given any $x \in U$, $\exists\ \epsilon > 0$ such that given any $y \in S$ with $d(x,y) < \epsilon$, $y \in U$

② A set $V \subseteq S$ is called closed if its complement $S \setminus V$ is an open set

③ $x \in S$ is called an interior point of S if there exist a neighborhood of x contained in S. If S is a metric space, then $x \in S$ is an interior pt if $\exists\, \epsilon > 0$ s.t $\forall y$ s.t $d(x,y) < \epsilon$, $y \in S$

The set of all interior pts of S form the interior of S. Thus, if S is a metric space:

$$\underbrace{\text{int}(S)}_{\text{interior of } S} = \{x \mid \exists\, \epsilon > 0 \text{ s.t } \forall y \text{ s.t } d(x,y) < \epsilon,\ y \in S\}$$

What can I say if interior$(C) = \phi$

eg: sufficient conditions

① $C \subseteq$ hyperplane. In particular, if S is affine this is necessary & sufficient

② eg: a shell

③ eg: $C = \partial K$ in topological space S (see next page for defn of boundary of set K denoted by ∂K)

④ The set of pts of a set S s.t: every neighborhood of a point from the set consists of atleast one point in S and one point not in S is called the boundary ∂S of S. If S is a metric space

$$\partial S = \{x \in S \mid \forall \epsilon > 0, \exists y \text{ s.t } d(x,y) < \epsilon \text{ \& } y \in S \text{ and } \exists y' \text{ s.t } d(x,y') < \epsilon \text{ \& } y' \notin S\}$$

⑤ Let S be a subset of a topological space X. A point $x \in X$ is a limit point of S if every neighborhood of x contains at least one point of S different from x itself.

can be relaxed to open neighborhoods without loss

If S happens to have an associated metric d, and $A \subseteq S$, then $\alpha \in S$ is a limit point of A iff:

$$\forall \epsilon > 0 : \{x \in A \text{ s.t } 0 < d(x, \alpha) < \epsilon\} \neq \phi$$

[Note: α need not belong to A]

Informally speaking, is a limit point of A if there are points in A that are different from but arbitrarily close to it

⑥ Closure of S [$cl(S)$] $= S \cup \{\text{limit points of } S\}$

Some standard results that we will regularly invoke for topological spaces

1. Intersection of (even uncountable) closed sets is closed
2. Union of (even uncountable) open sets is open
3. Intersection of finite number of open sets is open
4. Union of finite number of closed sets is closed
5. S is closed iff S^c is open

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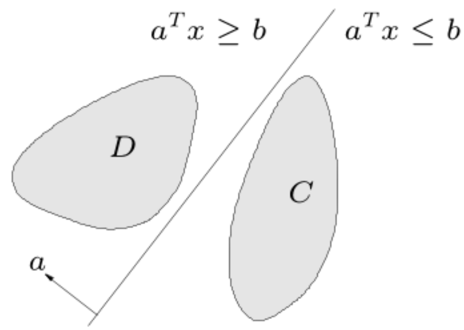


the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions (*e.g.*, C is closed, D is a singleton)

Proof: Let $S = \{x - y \mid x \in C, y \in D\}$.
Now we can prove (see http://www.cse.iitb.ac.in/~cs709/notes/eNotes/ExtraProblems-1.pdf, Q1) that S, being a sum of two convex sets, is convex.
Since $C \cap D = \phi$, $0 \notin S$.

(a) Suppose $0 \notin cl(S)$: Consider the sets $\{0\}$ and $cl(S)$. We will prove that $\exists\, a \neq 0$ s.t

$$\underbrace{a^T z > 0 \;\forall\, z \in cl(S)}_{\text{Q: How to choose } a\text{?}} \;\&\; \underbrace{a^T w = 0 \text{ for } w \in \{0\}}_{\text{obvious}}$$

Q: How to choose a?

complete proof given in class: H/W

i.e $\exists a$ s.t $a^T(x-y) > 0 \;\forall\; x-y \in S$

i.e $a^T x > a^T y \;\forall\; x \in C$ & $y \in D$

Let $b = \inf_{x \in C} a^T x$. Then we proved existence of a & b s.t

$$a^T x \geq b \;\forall\; x \in C \;\;\&\;\; a^T y \leq b \;\;\forall\; y \in D$$

(b) suppose $0 \in cl(S)$. Since $0 \notin S$, $0 \in$ bndry (S)

If interior $(S) = \phi$ (empty), S must be $\subseteq \{z \mid a^T z = b\}$ A hyperplane

& the hyperplane must includ 0 on bndry (S)

$\Rightarrow b = 0$. i.e $a^T x = a^T y \;\forall\; x \in C$ & $y \in D$

$\Rightarrow$ we have a trivial separating hyperplane limit by $\bar{a}$, we have

$$\boxed{a(\epsilon_k)^T z > 0} \quad \forall\, z \in S_{-\epsilon_k}$$

for all k & therefore

$$\boxed{\bar{a}^T z > 0 \quad \forall\, z \in \text{interior}(S)}$$

proof by contradiction

and

$$\bar{a}^T z \geq 0 \quad \forall\, z \in S$$

(use the property that a convex set is connected.)

that is

$$\bar{a}^T x \geq \bar{a}^T y$$

$$\forall\, x \in C \;\&\; y \in D$$

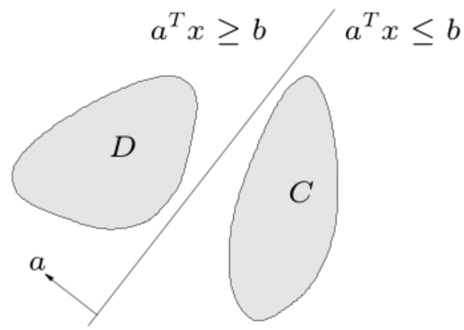
Hence proved!

Separating hyperplane theorem

Thus

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



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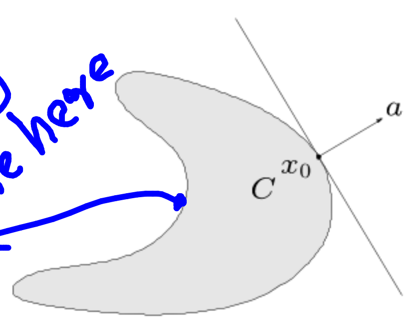
no supporting hyperplane here



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C