Definition 41 [Subgradient]: *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A vector $\mathbf{h} \in \Re^n$ is said to be a subgradient of f at the point $\mathbf{x} \in \mathcal{D}$ if*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{h}^T(\mathbf{y} - \mathbf{x})$$

for all $\mathbf{y} \in \mathcal{D}$. The set of all such vectors is called the subdifferential of f at $\mathbf{x}$.

Theorem 76 *Let $f : \mathcal{D} \rightarrow \Re$ be a convex function defined on a convex set $\mathcal{D}$. A point $\mathbf{x} \in \mathcal{D}$ corresponds to a minimum if and only if*

$$\nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \geq 0$$

for all $\mathbf{y} \in \mathcal{D}$.

If $\nabla f(\mathbf{x})$ is nonzero, it defines a supporting hyperplane to $\mathcal{D}$ at the point $\mathbf{x}$. Theorem 77 implies that for a differentiable convex function defined on an open set, every critical point must be a point of (global) minimum.

Theorem 77 *Let $f : \mathcal{D} \rightarrow \Re$ be differentiable and convex on an open convex domain $\mathcal{D} \subseteq \Re^n$. Then $\mathbf{x}$ is a critical point of f if and only if it is a (global) minimum.*

Theorem 78 *Let $f : \mathcal{D} \rightarrow \Re$ with $\mathcal{D} \subseteq \Re^n$ be differentiable on the convex set $\mathcal{D}$. Then,*

1. *f is convex on $\mathcal{D}$ if and only if is its gradient ∇f is monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$*

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T (\mathbf{x} - \mathbf{y}) \geq 0 \tag{4.53}$$

2. *f is strictly convex on $\mathcal{D}$ if and only if is its gradient ∇f is strictly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$ with $\mathbf{x} \neq \mathbf{y}$,*

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T (\mathbf{x} - \mathbf{y}) > 0 \tag{4.54}$$

3. *f is uniformly or strongly convex on $\mathcal{D}$ if and only if is its gradient ∇f is uniformly monotone. That is, for all $\mathbf{x}, \mathbf{y} \in \Re$,*

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T (\mathbf{x} - \mathbf{y}) \geq c||\mathbf{x} - \mathbf{y}||^2 \tag{4.55}$$

for some constant $c > 0$.

Necessity: Suppose f is uniformly convex on $\mathcal{D}$. Then from theorem 75, we know that for any $\mathbf{x}, \mathbf{y} \in \mathcal{D}$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) - \frac{1}{2}c||\mathbf{y} + \mathbf{x}||^2$$
$$f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla^T f(\mathbf{y})(\mathbf{x} - \mathbf{y}) - \frac{1}{2}c||\mathbf{x} + \mathbf{y}||^2$$

Adding the two inequalities, we get (4.55). If f is convex, the inequalities hold with $c = 0$, yielding (4.54). If f is strictly convex, the inequalities will be strict, yielding (4.54).

Sufficiency: Suppose ∇f is monotone. For any fixed $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, consider the function $\phi(t) = f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$. By the mean value theorem applied to $\phi(t)$, we should have for some $t \in (0, 1)$,

$$\phi(1) - \phi(0) = \phi'(t) \tag{4.56}$$

Letting $\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$, (4.56) translates to

$$f(\mathbf{y}) - f(\mathbf{x}) = \nabla^T f(\mathbf{z})(\mathbf{y} - \mathbf{x}) \tag{4.57}$$

Also, by definition of monotonicity of ∇f, (from (4.53)),

$$(\nabla f(\mathbf{z}) - \nabla f(\mathbf{x}))^T (\mathbf{y} - \mathbf{x}) = \frac{1}{t} (\nabla f(\mathbf{z}) - \nabla f(\mathbf{x}))^T (\mathbf{z} - \mathbf{x}) \geq 0 \tag{4.58}$$

Combining (4.57) with (4.58), we get,

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) &= (\nabla f(\mathbf{z}) - f(\mathbf{x}))^T (\mathbf{y} - \mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \\ &\geq \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \end{aligned} \tag{4.59}$$

By theorem 75, this inequality proves that f is convex. Strict convexity can be similarly proved by using the strict inequality in (4.58) inherited from strict monotonicity, and letting the strict inequality follow through to (4.59). For the case of strong convexity, from (4.55), we have

$$\begin{aligned} &\phi'(t) - \phi'(0) = (\nabla f(\mathbf{z}) - f(\mathbf{x}))^T (\mathbf{y} - \mathbf{x}) \\ &= \frac{1}{t} (\nabla f(\mathbf{z}) - f(\mathbf{x}))^T (\mathbf{z} - \mathbf{x}) \geq \frac{1}{t}c||\mathbf{z} - \mathbf{x}||^2 = ct||\mathbf{y} - \mathbf{x}||^2 \end{aligned} \tag{4.60}$$

$$\phi(1) - \phi(0) - \phi'(0) = \int_0^1 [\phi'(t) - \phi'(0)]dt \geq \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2 \tag{4.61}$$

which translates to

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla^T f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + \frac{1}{2}c||\mathbf{y} - \mathbf{x}||^2$$

What abt local maxima/minima & subgradient?

$\nabla f(x) = 0$ & f is convex then x is global min

What if $g_x = 0$?

$$f(y) \geq f(x) + g_x^T(y-x) \quad \forall y$$

if $g_x = 0$ then $f(y) \geq f(x) \Rightarrow x$ is pt of global min

Eg: $\min_x \frac{1}{2}\|y-x\|^2 + \lambda\|x\|_1$ (Regularizer) $\left(\underset{x}{\operatorname{argmin}} \|y-x\|^2 + \lambda\|x\|_1 = x^*\right)$

$\lambda \geq 0$

I will suggest a soln by setting "some" $g_x = 0$

$$x_i^* = \begin{cases} -\lambda + y_i & \text{if } y_i > \lambda \\ 0 & \text{if } -\lambda \leq y_i \leq \lambda \\ \lambda + y_i & \text{if } y_i < -\lambda \end{cases}$$

(*) Higher $\lambda \Rightarrow$ more x_i's are zeros

lots of zeros esp if several $|y_i| \leq \lambda$ .. sparsity

Why should this be imp for minimization? 2 ways of answering

① $g_x = \frac{1}{2}\nabla\left(\|y-x\|^2\right) + \lambda\partial\|x\|_1$

$= (x-y) + \lambda \begin{bmatrix} \text{sign}(x_1) \\ \vdots \\ \text{sign}(x_n) \end{bmatrix}$

② $\min_{x_i} \frac{1}{2}(y_i - x_i)^2 + \lambda|x_i|$ for each i

$g_{x_i} = (x_i - y_i) + \lambda\,\text{sign}(x_i)$

Subgradient method

subgradient method is simple algorithm to minimize nondifferentiable convex function f

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}$$

- $x^{(k)}$ is the kth iterate
- $g^{(k)}$ is **any** subgradient of f at $x^{(k)}$
- $\alpha_k > 0$ is the kth step size

not a descent method, so we keep track of best point so far

$$f_{\mathrm{best}}^{(k)} = \min_{i=1,\ldots,k} f(x^{(i)})$$

Step size rules

step sizes are fixed ahead of time

- *constant step size:* $\alpha_k = \alpha$ (constant)
- *constant step length:* $\alpha_k = \gamma / \|g^{(k)}\|_2$ (so $\|x^{(k+1)} - x^{(k)}\|_2 = \gamma$)
- *square summable but not summable:* step sizes satisfy

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

- *nonsummable diminishing:* step sizes satisfy

$$\lim_{k \to \infty} \alpha_k = 0, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$