Q: $(S^n_{++})^* = ?$ $\quad \text{int}(S^n_+) = S^n_{++}$

Ans: $(S^n_{++})^* = S^n_+$ (will be done formally for general case of convex cones)

C = convex cone

$C^{**} = \text{cl}(C)$

Q: Consider an application of psd cone for optimization. (thru LP)

↳ We will first see (weak) duality in a linear optimization problem (LP)

↳ Next we look at generalized (conic) inequalities and properties that the cone must satisfy for the inequality to be a valid inequality

↳ Next we generalize LP to conic program (CP) using generalized inequality and realize weak duality for CP thru dual cones

LP: $\min c^T x$

s.t $Ax \geq b$

$x \in \mathbb{R}^n$

LD: $\max \lambda^T b$

s.t $A^T \lambda = c$

$\lambda \in \mathbb{R}^m_+$

Generalizations could be by

Generalizing the objective to non-linear (convex) objectives

$\|c - x\|^2$ or $x^T Q x + b$

Generalizing constraints to be nonlinear constraints

$\|x - c\|^2 \leq t$

Generalize the notion of inequality itself

This generalization is as powerful as others

Consider

linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b} \le 0 \end{array}$$

Conic Program (CP)

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b} \le_K 0 \end{array}$$

Let: $\lambda \geq 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b) \le 0$

$$\Rightarrow c^Tx \geq c^Tx + \lambda^T(-Ax+b)$$
$$= \lambda^Tb + (c-A^T\lambda)^Tx$$
$$\geq \min_x \lambda^Tb + (c-A^T\lambda)^Tx$$
$$= \begin{cases} \lambda^Tb & \text{if } A^T\lambda = c \\ -\infty & \text{if } A^T\lambda \neq c \end{cases}$$

independent of λ

independent of x

$$\Rightarrow \boxed{\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} c^Tx & \max\limits_{\lambda\geq 0} b^T\lambda \\ \text{s.t } Ax\geq b & \geq \quad \text{s.t } A^T\lambda = c \end{array}}$$

Primal LP (lower bounded)

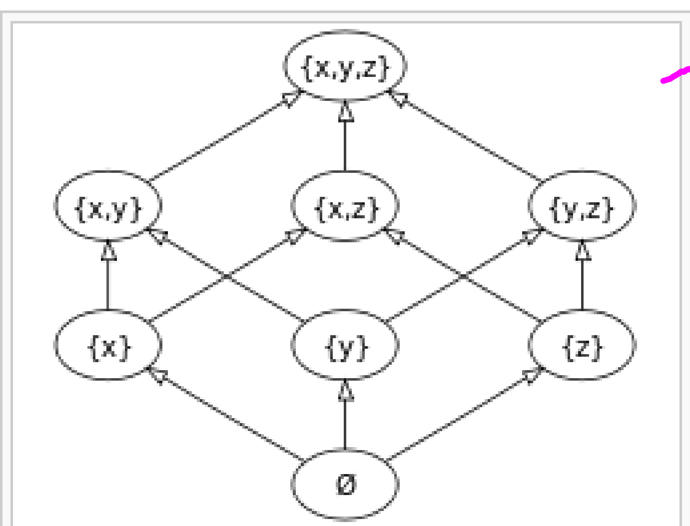
Dual LP (upper bounded)

- Q: How to generalise $-Ax+b\le 0$ to $-Ax+b \le_K 0$ s.t $\le_K$ is a generalised inequality & K some set?

- what properties should K satisfy so that $\le_K$ satisfies properties of generalized inequalities?

To prove that K being convex cone & pointed are necessary & sufficient conditions for $\succeq_K$ to be a valid inequality, recall that any partial order $\succeq_K$ should satisfy the following properties (refer page 51 of www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf i.e Section 1.4.1)

1. *Reflexivity:* $a \geq a$;
2. *Anti-symmetry:* if both $a \geq b$ and $b \geq a$, then $a = b$;
3. *Transitivity:* if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. *Compatibility with linear operations:*
   (a) *Homogeneity:* if $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$ ("One can multiply both sides of an inequality by a nonnegative real")
   (b) *Additivity:* if both $a \geq b$ and $c \geq d$, then $a + c \geq b + d$ ("One can add two inequalities of the same sign").



The Hasse diagram of the set of all subsets of a three-element set {x, y, z}, ordered by inclusion.

example partial order $\subseteq$ over sets (source: http://en.wikipedia.org/wiki/Partially_ordered_set)

that is, the $\subseteq$ partial order

Proof:

(a) K being pointed convex cone $\Rightarrow$ $\succcurlyeq_K$ is a partial order

(i) $\underbrace{a \succcurlyeq_K a}_{\text{reflexivity}}$ since $a-a=0 \in K$ ($\because$ K is cone)

(2) If $a \succcurlyeq_K b$ & $b \succcurlyeq_K a$ then $a=b$ } anti-symmetry

since

$a-b \in K$ and $b-a \in K \Rightarrow b-a=0$

($\because$ K is pointed)

(3) If both $a \succcurlyeq_K b$ and $b \succcurlyeq_K c$ then $\underbrace{a \succcurlyeq_K c}_{\text{Transitivity}}$

since

$a-b \in K$ and $b-c \in K \Rightarrow (a-b)+(b-c) \in K$

i.e. $a-c \in K$

($\because$ K is a convex cone)

(4) (a) If $a \succcurlyeq_K b$ and $\lambda \geq 0$ then $\underbrace{\lambda a \succcurlyeq_K \lambda b}_{\text{Homogeneity}}$

since

if $a-b \in K$ & $\lambda \geq 0$ then $\lambda(a-b) \in K$

($\because$ K is a cone)

(b) If both $a \succcurlyeq_K b$ & $c \succcurlyeq_K d$ then $\underbrace{a+c \succcurlyeq_K b+d}_{\text{Additivity}}$

since

If both $a-b \in K$ & $c-d \in K$ then

$$(a-b) + (c-d) = (a+c) - (b+d) \in K$$

($\because$ K is a convex cone)

(b) $\succeq_K$ is a partial order $\Rightarrow$ K is a pointed convex cone

① If $x, y \in K$ then $\theta_1 x + \theta_2 y \in K$

$\forall \theta_1, \theta_2 \geq 0$ (K is a convex cone)

since

if $x \succeq_K 0$ & $y \succeq_K 0$ then $\theta_1 x \succeq_K 0 \; \forall \theta_1 \geq 0$

and $\theta_2 y \succeq_K 0 \; \forall \theta_2 \geq 0$ (Homogeneity of $\succeq_K$)

and thus $\theta_1 x + \theta_2 y \succeq_K 0$ (Additivity of $\succeq_K$)

② If $x \in K$ & $-x \in K$ then $x = 0$

(K is pointed)

since

if $x \succeq_K 0$ & $-x \succeq_K 0$ then

$0 \succeq_K x$ ($x \succeq_K x$ by reflexivity and adding $x \succeq_K x$ & $-x \succeq_K 0$ by additivity) and $-x \succeq_K x$ ($-x \succeq_K 0$ & $0 \succeq_K x$ just derived and adding... by additivity)

and similarly $x \succeq_K -x$ (similar use of additivity & reflexivity)

and $-x \succeq_K x$ & $x \succeq_K -x \Rightarrow x = -x$ (by anti-symmetry)

which is $x + x = 2x = 0$ ie $x = 0$

Questions: (Additional properties over & above K being pointed convex cone)

① Suppose $a^i \succeq_K b^i \ \forall i$ & $a^i \to a$ & $b^i \to b$

Then for $a \succeq_K b$ what more is reqd of K?

↳ Ans: Necessary condition is that

$a^i - b^i \to a - b \in K$

ie K is closed (Also happens to be a sufficient condition)

② What is reqd so that $\exists\, a \succ_K b$ (ie $b \not\succeq_K a$)?

↳ Ans: Sufficient condition is that $a - b \in \text{int}(K)$

(H/W) ie $\text{int}(K) \neq \emptyset$ OR K has non-empty interior

We will motivate through linear programming (LP) the concept of generalised inequalities:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to } -Ax+b \le 0$$

→ LINEAR PROGRAM

→ Can be rewritten as $Ax \ge b$ or $Ax - b \in \mathbb{R}^n_+$

Note: $\mathbb{R}^n_+$ is a CONE. How abt defining generalised inequality for a cone K as: $c \succeq_K d$ iff $c - d \in K$ and a general conic program as:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to } -Ax+b \preceq_K 0$$

→ CONIC PROGRAM

→ That is, $Ax - b \in K$

K is a proper cone

Generalized inequalities

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

→ Also referred to as a regular cone

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

} Some restrictions on K that we will require. H/w: WHY?

i.e. K has no st. lines passing thru 0

↳ i.e if $a, -a \in K$, then $a = 0$

examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \ge 0, i = 1, \dots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \ge 0 \text{ for } t \in [0, 1]\}$$

Q: What if $n \to \infty$ ... can you get proper cones under additional constraints?

Consider

linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\min_{x\in\Re^n} \; c^T x \quad \text{subject to} \quad -Ax+b \le 0$$

Conic Program (CP)

$$\min_{x\in\Re^n} \; c^T x \quad \text{subject to} \quad -Ax+b \le_K 0$$

K is a regular/proper cone

Let: $\lambda \ge 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b) \le 0$

$$\Rightarrow c^T x \ge c^T x + \lambda^T(-Ax+b)$$

$$= \lambda^T b + (c - A^T\lambda)^T x$$

$$\ge \min_x \lambda^T b + (c - A^T\lambda)^T x$$

$$= \begin{cases} \lambda^T b & \text{if } A^T\lambda = c \\ -\infty & \text{if } A^T\lambda \ne c \end{cases}$$

independent of λ

independent of x

$$\Rightarrow \quad \min_{x\in\mathbb{R}^n} c^T x \;\; \text{s.t} \;\; Ax \ge b \quad \ge \quad \max b^T\lambda \;\; \lambda \ge 0 \;\; \text{s.t} \;\; A^T\lambda = c$$

Primal LP (lower bounded)

Dual LP (upper bounded)

Generalised cone program

$$\min_{x\in V} \; \langle c, x\rangle_V \quad \text{subject to} \quad Ax - b \in K$$

We need an equivalent $\lambda \in D \supseteq K^*$ s.t

$$\langle \lambda, Ax-b\rangle \ge 0$$

This K^* s.t

$$D = \{\lambda \mid \langle \lambda, Ax-b\rangle \ge 0, x \in V \;\; \forall Ax-b \in K\}$$

& $D \supseteq K^*$ is DUAL CONE of K!

by dual)

by primal)

Called the weak duality theorem for Linear Program

$K_* = \{\lambda : \lambda^T \xi \geq 0 \,\forall \xi \in K\}$ is the cone dual to K

[defn on page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

With this, follows weak duality theorem for CONIC PROGRAM

$$\min_{x \in V} \langle c, x\rangle \;\; \text{s.t. } Ax \geq_K b \quad \geq \quad \max_{\lambda \in K^*} \langle b, \lambda\rangle \;\; \text{s.t. } A^T\lambda = c$$

$\boxed{CP}$ Primal CP (lower bounded by dual)

$\boxed{CD}$ Dual CP (upperbounded by primal)

Note: ① Both LP & CP dealt with affine objective

② CP dealt with the generalised conic inequalities

③ Later, in convex programs, we will deal with the more general convex functions in the objective

Notes:

① If $K = \mathbb{R}^n_+$, the CP is an LP

If $K = S^n_+$, the CP is an SDP

Set of all $n \times n$ symmetric positive semidefinite matrices

Semi-definite program

② Any generic convex program can be expressed as a cone program (CP)

① If K is a closed convex cone then $K^{**} = K$

more generally $K^{**} = \text{closure}(K)$ (abbreviated as $\text{cl}(K)$)

if K is just a convex cone

Proof: We will prove that if K is closed then

$K^{**} = K$

ⓐ $K \subseteq K^{**}$ since $x \in K \Rightarrow \langle x, y \rangle \geq 0 \; \forall y \in K^{*}$

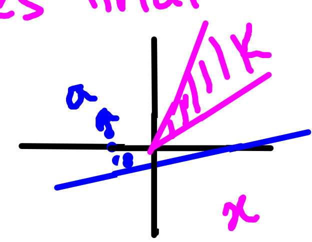
$\Rightarrow x \in K^{**}$

ⓑ $K^{**} \subseteq K$ ... We will prove by contradiction

Suppose $x \in K^{**}$ but $x \notin K$

↳ K^{**} is closed since any dual cone is intersection of half spaces that are closed

↳ $\{x\}$ is a singleton set



$\Rightarrow$ By "strict separating hyperplane theorem" (on next page and proved later)

$\exists \, a \in V \; \& \; b \in \mathbb{R}$ s.t. $\langle a, x \rangle < b \; \& \; \langle a, y \rangle \geq b \; \forall y \in K$

claim: $b = 0$ if V is a closed convex cone

(since $y = 0 \in K^{**}$)

$\Rightarrow \langle a, x \rangle < 0 \leq \langle a, y \rangle \; \forall y \in K$

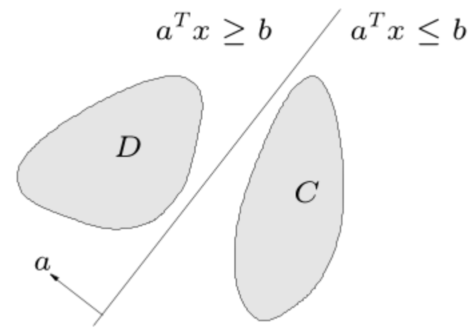
$\Rightarrow a \in K^{*} \; \& \; \therefore x \notin K^{**}$ [contradiction]

A fundamental thm

Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separating hyperplane theorem

strict separation requires additional assumptions (*e.g.*, C is closed, D is a singleton)

consequence

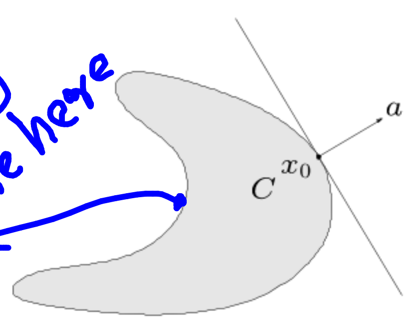
Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

no supporting hyperplane here



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

(2) In fact, if K is a proper cone then
K^* is also proper

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

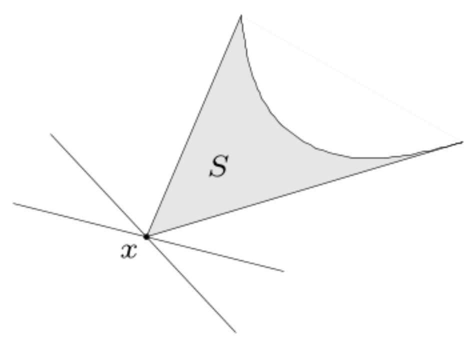
dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

Minimum and minimal elements via dual inequalities

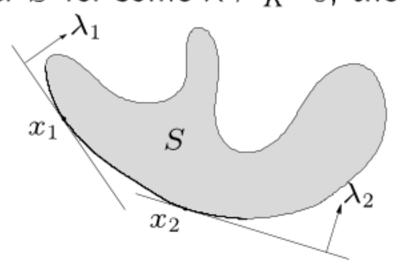
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

FROM DUAL OF NORM CONE TO DUAL NORM

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$

The dual of $K = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \le t\}$

is $K^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \le v\}$

Where $\|u\|_* = \sup\{u^T x \mid \|x\| \le 1\}$

Proof: We need to show that

$$x^T u + tv \ge 0 \text{ whenever } \|x\| \le t \iff \|u\|_* \le v. \tag{2.20}$$

1) Let us start by showing that the righthand condition on (u, v) implies the lefthand condition. Suppose $\|u\|_* \le v$, and $\|x\| \le t$ for some $t > 0$. (If $t = 0$, x must be zero, so obviously $u^T x + vt \ge 0$.) Applying the definition of the dual norm, and the fact that $\|-x/t\| \le 1$, we have

$$u^T(-x/t) \le \|u\|_* \le v,$$

and therefore $u^T x + vt \ge 0$.

2) Next we show that the lefthand condition in (2.20) implies the righthand condition in (2.20). Suppose $\|u\|_* > v$, *i.e.*, that the righthand condition does not hold. Then by the definition of the dual norm, there exists an x with $\|x\| \le 1$ and $x^T u > v$. Taking $t = 1$, we have

$$u^T(-x) + v < 0,$$

which contradicts the lefthand condition in (2.20).

(Proof from Boyd)