Optimization Principles for Univariate Functions

Critical Point and Local Extreme Value

Given a critical point c, the following test helps determine if $f(c)$ is a local extreme value:

Procedure

[Local Extreme Value]: *Let c be an isolated critical point of f*

1. *$f(c)$ is a local minimum if $f(x)$ is decreasing in an interval $[c-\epsilon_1, c]$ and increasing in an interval $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ is increasing in an interval $[c-\epsilon_1, c]$ and decreasing in an interval $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*

Given a critical point c, *first derivative test* (sufficient condition) helps determine if $f(c)$ is a local extreme value:

Procedure

[First derivative test]: *Let c be an isolated critical point of f*

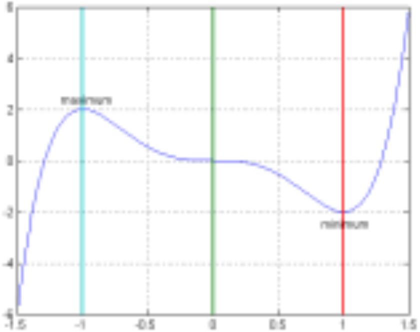
1. *$f(c)$ is a local minimum if the sign of $f'(x)$ changes from negative in $[c-\epsilon_1, c]$ to positive in $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ the sign of $f'(x)$ changes from positive in $[c-\epsilon_1, c]$ to negative in $[c, c+\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
3. *If $f'(x)$ is positive in an interval $[c-\epsilon_1, c]$ and also positive in an interval $[c, c-\epsilon_2]$, or $f'(x)$ is negative in an interval $[c-\epsilon_1, c]$ and also negative in an interval $[c, c-\epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$, then $f(c)$ is not a local extremum.*

First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has

First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has the derivative $f'(x) = 15x^2(x+1)(x-1)$. The critical points are 0, 1 and -1. Of the three, the sign of $f'(x)$ changes at 1 and -1, which are local minimum and maximum respectively. The sign does not change at 0, which is therefore not a local supremum.



First Derivative Test: Critical Point and Local Extreme Value

As another example, consider the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then,

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Note that $f(x)$ is discontinuous at $x = 0$, and therefore $f'(x)$ is not defined at $x = 0$. All numbers $x \geq 0$ are critical numbers. $f(0) = 0$ is a local minimum, whereas $f(x) = 1$ is a local minimum as well as a local maximum $\forall x > 0$.

Strict Convexity and Extremum

- A differentiable function f is said to be *strictly convex* (or *strictly concave up*) on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is increasing on $\mathcal{I}$.
- Recall from theorem 7, the graphical interpretation of the first derivative $f'(x)$; $f'(x) > 0$ implies that $f(x)$ is increasing at x.
- Similarly, $f'(x)$ is increasing when $f''(x) > 0$. This gives us a sufficient condition for the strict convexity of a function:

Claim

If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) > 0, \ \forall x \in \mathcal{I}$, then the slope of the function is always increasing with x and the graph is strictly convex. This is illustrated in Figure 8.

- On the other hand, if the function is strictly convex and doubly differentiable in $\mathcal{I}$, then $f''(x) \geq 0, \ \forall x \in \mathcal{I}$.

Strict Convexity and Extremum (Illustrated)

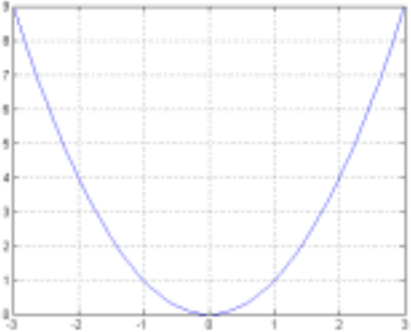


Figure 8:

Strict Convexity and Extremum: Slopeless interpretation (SI)

Claim

A function f is strictly convex on an open interval $\mathcal{I}$, iff

actual function eval

$$f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2) \tag{1}$$

whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.

interpolation using line segment

SI: Necessity when f is differentiable

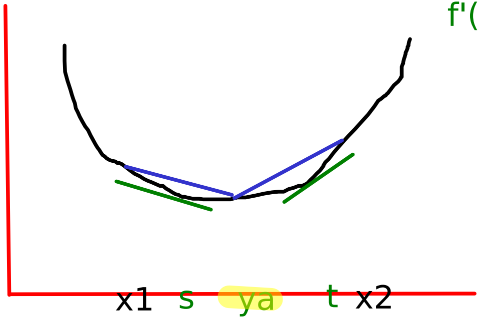
First we will prove the **necessity**.

[2]For the case $x_2 < x_1$, the proof is very similar.

SI: Necessity when f is differentiable

First we will prove the **necessity**. Suppose f' is increasing on $\mathcal{I}$. Let $0 < a < 1$, $x_1, x_2 \in \mathcal{I}$ and $x_1 \neq x_2$. Without loss of generality assume that[2] $x_1 < x_2$. Then, $x_1 < ax_1 + (1-a)x_2 < x_2$ and therefore $ax_1 + (1-a)x_2 \in \mathcal{I}$. By the mean value theorem, there exist s and t with $x_1 < s < ax_1 + (1-a)x_2 < t < x_2$, such that $f(ax_1 + (1-a)x_2) - f(x_1) = f'(s)(x_2 - x_1)(1-a)$ and $f(x_2) - f(ax_1 + (1-a)x_2) = f'(t)(x_2 - x_1)a$. Therefore,

f'(s) < f'(t)



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$$\begin{aligned} (1-a)f(x_1) - f(ax_1 + (1-a)x_2) + af(x_2) &= \\ a\left[f(x_2) - f(ax_1 + (1-a)x_2)\right] - (1-a)\left[f(ax_1 + (1-a)x_2) - f(x_1)\right] &= \\ a(1-a)(x_2 - x_1)\left[f'(t) - f'(s)\right] & \end{aligned}$$

Since $f(x)$ is strictly convex on $\mathcal{I}$, $f'(x)$ is increasing on $\mathcal{I}$ and therefore, $f'(t) - f'(s) > 0$. Moreover, $x_2 - x_1 > 0$ and $0 < a < 1$. Thus, $(1-a)f(x_1) - f(ax_1 + (1-a)x_2) + af(x_2) > 0$, or equivalently, $f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2)$, which is what we wanted to prove in inequality (1).

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Suppose the inequality in inqeuality (1) holds. Then, taking limits,

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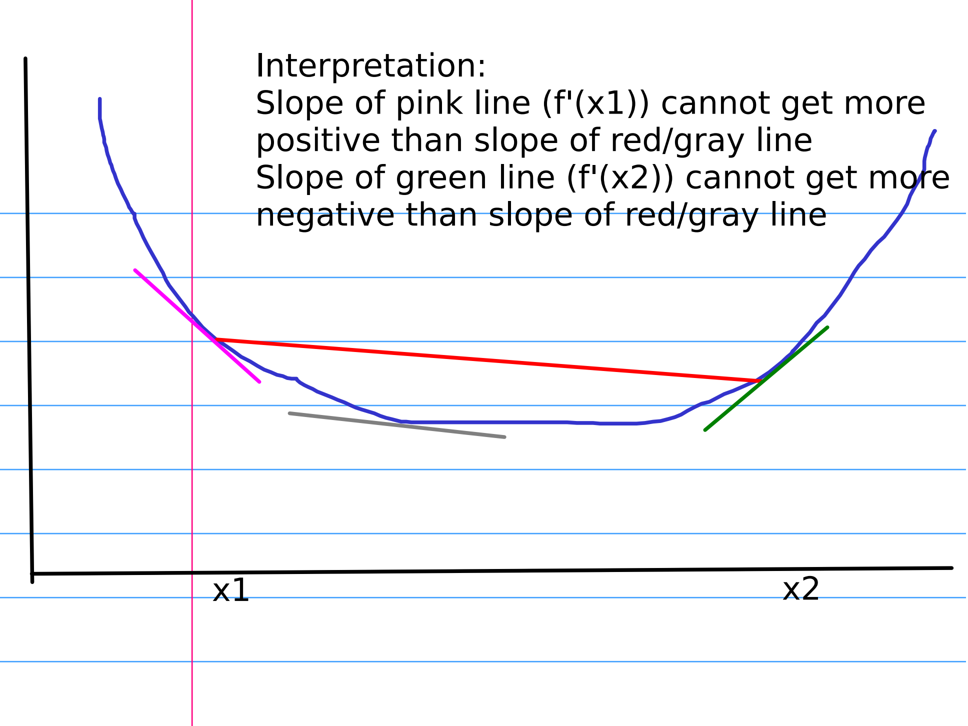
$$f'(x_2)(x_1 - x_2) \leq f(x_1) - f(x_2) \quad (2)$$

Similarly, we can show that

$$f'(x_1)(x_2 - x_1) \leq f(x_2) - f(x_1) \quad (3)$$

Adding the left and right hand sides of inequalities in (2) and (3), and multiplying the resultant inequality by -1 gives us


Interpretation:
Slope of pink line (f'(x1)) cannot get more positive than slope of red/gray line
Slope of green line (f'(x2)) cannot get more negative than slope of red/gray line
x1
x2

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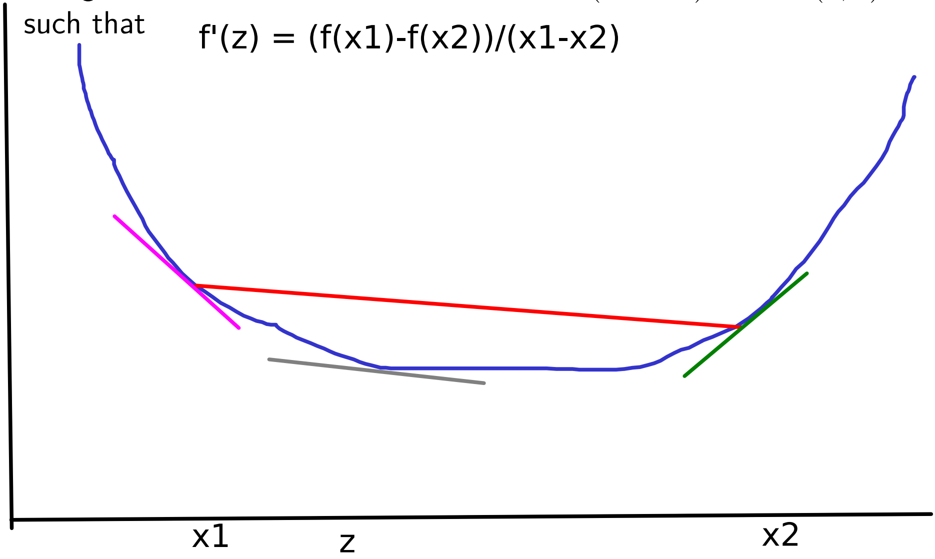
$$\left(f'(x_2) - f'(x_1)\right)(x_2 - x_1) \geq 0 \quad (4)$$

We now need to prove that the inequality in (4) is strict.

SI: Sufficiency when f is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

f'(z) = (f(x1)-f(x2))/(x1-x2)



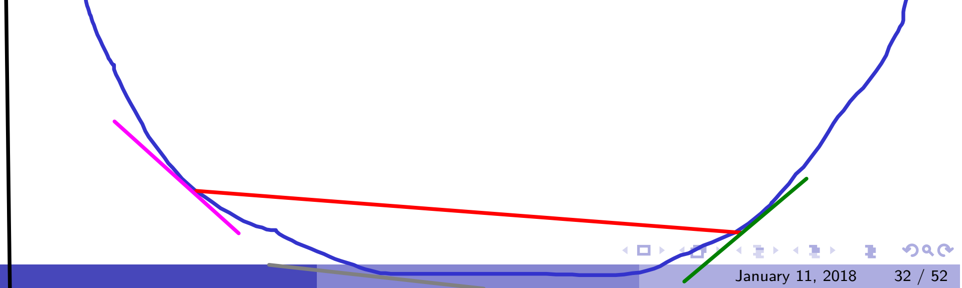
SI: Sufficiency when f is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0,1)$ such that

$$f(x_2) - f(x_1) = f'(z)(x_2 - x_1) \tag{5}$$

Since (4) holds for any $x_1, x_2 \in \mathcal{I}$, it also hold for $x_2 = z$. Therefore,

f'(z) >= f'(x1)
f'(z) <=f'(x2)



SI: Sufficiency when f is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

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Since (4) holds for any $x_1, x_2 \in \mathcal{I}$, it also hold for $x_2 = z$. Therefore,

$$(f'(z) - f'(x_1))(x_2 - x_1) = \frac{1}{t}(f'(z) - f'(x_1))(z - x_1) \geq 0$$

Additionally using (5), we get

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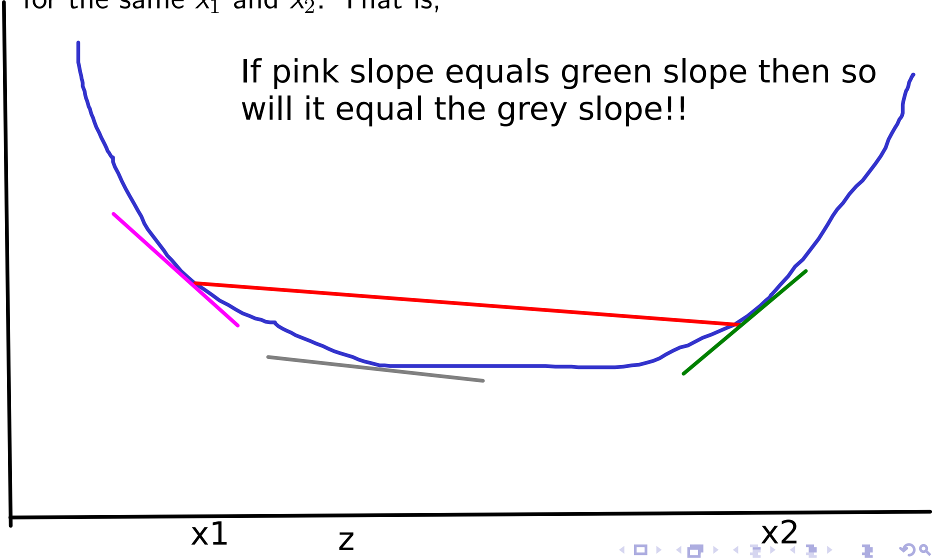
Additionally using (5), we get

$$f(x_2) - f(x_1) = (f'(z) - f'(x_1))(x_2 - x_1) + f'(x_1)(x_2 - x_1) \geq f'(x_1)(x_2 - x_1) \tag{6}$$

SI: Sufficiency when f is differentiable (contd)

Suppose equality holds in (4) for some $x_1 \neq x_2$. Then it holds in (6) for the same x_1 and x_2. That is,



SI: Sufficiency when f is differentiable (contd)

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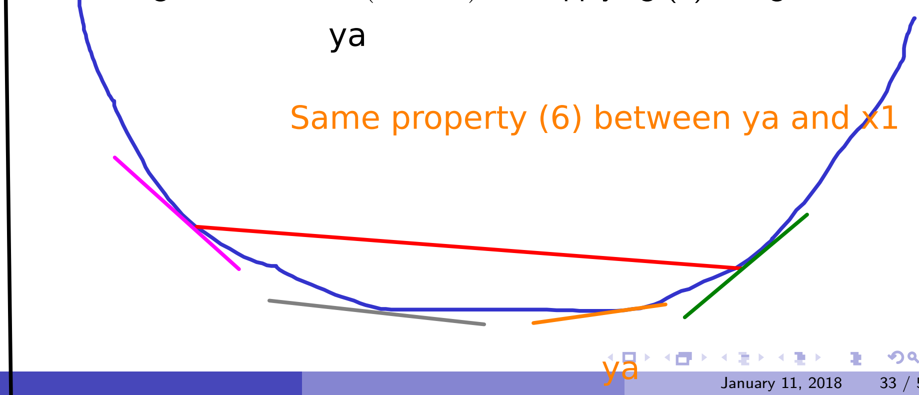
$$f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1) \quad (7)$$

Substituting x_2 with $x_1 + a(x_2 - x_1)$ and applying (6), we get



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Substituting x_2 with $x_1 + a(x_2 - x_1)$ and applying (6), we get

$$f(x_1) + af'(x_1)(x_2 - x_1) \leq f(x_1 + a(x_2 - x_1)) \tag{8}$$

Further using (1) and 7, we can derive that

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Further using (1) and 7, we can derive that

Linear interpolation

$$f(x_1 + a(x_2 - x_1)) < (1 - a)f(x_1) + af(x_2) = f(x_1) + af'(x_1)(x_2 - x_1) \tag{9}$$

Contradiction between (8) and (9)!

SI: Sufficiency when f is differentiable (contd)

Thus, equations 8 and 9 contradict each other.

$$f(x_1) + af'(x_1)(x_2 - x_1) \leq f(x_1 + a(x_2 - x_1))$$

$$f(x_1 + a(x_2 - x_1)) < (1 - a)f(x_1) + af(x_2) = f(x_1) + af'(x_1)(x_2 - x_1)$$

Therefore, equality in 4 cannot hold for any $x_1 \neq x_2$, implying that

$$\left(f'(x_2) - f'(x_1)\right)(x_2 - x_1) > 0$$

that is, $f'(x)$ is increasing and therefore f is convex on $\mathcal{I}$. □

Strict Concavity

- A differentiable function f is said to be *strictly concave* on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is decreasing on $\mathcal{I}$.
- Recall from theorem 7, the graphical interpretation of the first derivative $f'(x)$; $f'(x) < 0$ implies that $f(x)$ is decreasing at x.
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- Similarly, $f'(x)$ is (strictly) monotonically decreasing when $f''(x) > 0$. This gives us a sufficient condition for the concavity of a function:

Claim

If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) < 0$, $\forall x \in \mathcal{I}$, then the slope of the function is always decreasing with x and the graph is strictly concave.

Strict Concavity

On the other hand, if the function is strictly concave and doubly differentiable in $\mathcal{I}$, then $f''(x) \leq 0, \ \forall x \in \mathcal{I}$. This is illustrated in Figure 9.

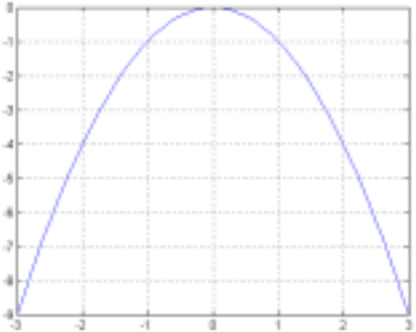


Figure 9:

Strict Concavity (slopeless interpretation)

There is also a slopeless interpretation of concavity as stated in the following theorem:

Claim

A differentiable function f is strictly concave on an open interval $\mathcal{I}$, iff

$$f(ax_1 + (1-a)x_2) > af(x_1) + (1-a)f(x_2) \qquad (10)$$

whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.

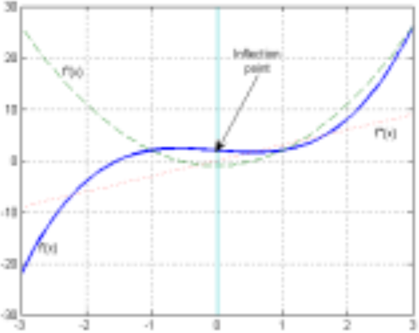
The proof is similar to that for theorem 12.

Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$.

Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$. It's slope decreases as x increases to 0 ($f''(x) < 0$) and then the slope increases beyond $x = 0$ ($f''(x) > 0$). The point 0, where the $f''(x)$ changes sign is called the *inflection point*; the graph is strictly concave for $x < 0$ and strictly convex for $x > 0$. See Figure 10.



Convex & Concave Regions and Inflection Point

Along similar lines, study the function
$f(x) = \frac{1}{20}x^5 - \frac{7}{12}x^4 + \frac{7}{6}x^3 - \frac{15}{2}x^2$.

Convex & Concave Regions and Inflection Point

Along similar lines, study the function
$f(x) = \frac{1}{20}x^5 - \frac{7}{12}x^4 + \frac{7}{6}x^3 - \frac{15}{2}x^2$.
It is strictly concave on $(-\infty, -1]$ and $[3, 5]$ and strictly convex on $[-1, 3]$ and $[5, \infty]$.
The inflection points for this function are at $x = -1$, $x = 3$ and $x = 5$.

First Derivative Test: Restated using Strict Convexity

The *first derivative test* for local extrema can be restated in terms of strict convexity and concavity of functions.

First Derivative Test: Restated using Strict Convexity

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Procedure

[First derivative test in terms of strict convexity]: *Let c be a critical number of f and $f'(c) = 0$. Then,*

1. *$f(c)$ is a local minimum if the graph of $f(x)$ is strictly convex on an open interval containing c.*
2. *$f(c)$ is a local maximum if the graph of $f(x)$ is strictly concave on an open interval containing c.*

Strict Convexity: Restated using Second Derivative

If the second derivative $f''(c)$ exists, then the strict convexity conditions for the critical number can be stated in terms of the sign of of $f''(c)$, making use of previous results. This is called the *second derivative test*.

Strict Convexity: Restated using Second Derivative

If the second derivative $f''(c)$ exists, then the strict convexity conditions for the critical number can be stated in terms of the sign of of $f''(c)$, making use of previous results. This is called the *second derivative test*.

Procedure

[Second derivative test]: *Let c be a critical number of f where $f'(c) = 0$ and $f''(c)$ exists.*

1. *If $f''(c) > 0$ then $f(c)$ is a local minimum.*
2. *If $f''(c) < 0$ then $f(c)$ is a local maximum.*
3. *If $f''(c) = 0$ then $f(c)$ could be a local maximum, a local minimum, neither or both. That is, the test fails.*

Convexity, Minima and Maxima: Illustrations

Study the functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$:

Convexity, Minima and Maxima: Illustrations

Study the functions $f(x) = x^4$, $f(x) = -x^4$ and $f(x) = x^3$:

- If $f(x) = x^4$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is a local minimum.
- If $f(x) = -x^4$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is a local maximum.
- If $f(x) = x^3$, then $f'(0) = 0$ and $f''(0) = 0$ and we can see that $f(0)$ is neither a local minimum nor a local maximum. $(0, 0)$ is an inflection point in this case.

Convexity, Minima and Maxima: Illustrations (contd.)

Study the functions: $f(x) = x + 2\sin x$ and $f(x) = x + \frac{1}{x}$:

- If $f(x) = x + 2\sin x$, then $f'(x) = 1 + 2\cos x$. $f'(x) = 0$ for $x = \frac{2\pi}{3}, \frac{4\pi}{3}$, which are the critical numbers. $f''\left(\frac{2\pi}{3}\right) = -2\sin\frac{2\pi}{3} = -\sqrt{3} < 0 \Rightarrow f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum value. On the other hand, $f''\left(\frac{4\pi}{3}\right) = \sqrt{3} > 0 \Rightarrow f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum value.
- If $f(x) = x + \frac{1}{x}$, then $f'(x) = 1 - \frac{1}{x^2}$. The critical numbers are $x = \pm 1$. Note that $x = 0$ is not a critical number, even though $f'(0)$ does not exist, because 0 is not in the domain of f. $f''(x) = \frac{2}{x^3}$. $f''(-1) = -2 < 0$ and therefore $f(-1) = -2$ is a local maximum. $f''(1) = 2 > 0$ and therefore $f(1) = 2$ is a local minimum.

Global Extrema on Closed Intervals

Recall the extreme value theorem. A consequence is that:

- if either of c or d lies in (a, b), then it is a critical number of f;
- else each of c and d must lie on one of the boundaries of $[a, b]$.

This gives us a procedure for finding the maximum and minimum of a continuous function f on a closed bounded interval $\mathcal{I}$:

Procedure

[Finding extreme values on closed, bounded intervals]:

1. *Find the critical points in* $int(\mathcal{I})$.
2. *Compute the values of f at the critical points and at the endpoints of the interval.*
3. *Select the least and greatest of the computed values.*

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,
  - We first compute $f'(x) = 12x^2 - 16x + 5$ which is 0 at $x = \frac{1}{2}, \frac{5}{6}$.
  - Values at the critical points are $f(\frac{1}{2}) = 1$, $f(\frac{5}{6}) = \frac{25}{27}$.
  - The values at the end points are $f(0) = 0$ and $f(1) = 1$.
  - Therefore, the minimum value is $f(0) = 0$ and the maximum value is $f(1) = f(\frac{1}{2}) = 1$.

Global Extrema on Closed Intervals (contd)

- To compute the maximum and minimum values of $f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$,
  - We first compute $f'(x) = 12x^2 - 16x + 5$ which is 0 at $x = \frac{1}{2}, \frac{5}{6}$.
  - Values at the critical points are $f(\frac{1}{2}) = 1$, $f(\frac{5}{6}) = \frac{25}{27}$.
  - The values at the end points are $f(0) = 0$ and $f(1) = 1$.
  - Therefore, the minimum value is $f(0) = 0$ and the maximum value is $f(1) = f(\frac{1}{2}) = 1$.
- In this context, it is relevant to discuss the one-sided derivatives of a function at the endpoints of the closed interval on which it is defined.

Global Extrema on Closed Intervals (contd)

Definition

[One-sided derivatives at endpoints]: *Let f be defined on a closed bounded interval* $[a, b]$. *The (right-sided) derivative of f at* $x = a$ *is defined as*

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Similarly, the (left-sided) derivative of f at $x = b$ *is defined as*

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

Essentially, each of the one-sided derivatives defines one-sided slopes at the endpoints.

Global Extrema on Closed Intervals (contd)

Based on these definitions, the following result can be derived.

Claim

If f is continuous on $[a, b]$ and $f'(a)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at a.

- *If $f(a)$ is the maximum value of f on $[a, b]$, then $f'(a) \leq 0$ or $f'(a) = -\infty$.*
- *If $f(a)$ is the minimum value of f on $[a, b]$, then $f'(a) \geq 0$ or $f'(a) = \infty$.*

If f is continuous on $[a, b]$ and $f'(b)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions

Global Extrema on Closed Intervals (contd)

Based on these definitions, the following result can be derived.

Claim

If f is continuous on $[a, b]$ and $f'(a)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at a.

- *If $f(a)$ is the maximum value of f on $[a, b]$, then $f'(a) \leq 0$ or $f'(a) = -\infty$.*
- *If $f(a)$ is the minimum value of f on $[a, b]$, then $f'(a) \geq 0$ or $f'(a) = \infty$.*

If f is continuous on $[a, b]$ and $f'(b)$ exists as a real number or as $\pm\infty$, then we have the following necessary conditions for extremum at b.

- *If $f(b)$ is the maximum value of f on $[a, b]$, then $f'(b) \geq 0$ or $f'(b) = \infty$.*
- *If $f(b)$ is the minimum value of f on $[a, b]$, then $f'(b) \leq 0$ or $f'(b) = -\infty$.*

Global Extrema on Closed Intervals (contd)

The following result gives a useful procedure for finding **extrema on closed intervals**.

Claim

If f is continuous on $[a, b]$ and $f'(x)$ exists for all $x \in (a, b)$. Then,

- *If $f'(x) \leq 0, \ \forall x \in (a, b)$, then the minimum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical point $c \in (a, b)$, then $f(c)$ is the maximum value of f on $[a, b]$.*
- *If $f'(x) \geq 0, \ \forall x \in (a, b)$, then the maximum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical point $c \in (a, b)$, then $f(c)$ is the minimum value of f on $[a, b]$.*

Global Extrema on Open Intervals

The next result is very useful for finding **extrema on open intervals.**

Claim

Let $\mathcal{I}$ be an open interval and let $f'(x)$ exist $\forall x \in \mathcal{I}$.

- *If $f''(x) \geq 0, \ \forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global minimum value of f on $\mathcal{I}$.*
- *If $f''(x) \leq 0, \ \forall x \in \mathcal{I}$, and if there is a number $c \in \mathcal{I}$ where $f'(c) = 0$, then $f(c)$ is the global maximum value of f on $\mathcal{I}$.*

For example, let $f(x) = \frac{2}{3}x - \sec x$ and $\mathcal{I} = (\frac{-\pi}{2}, \frac{\pi}{2})$.
$f'(x) = \frac{2}{3} - \sec x \tan x = \frac{2}{3} - \frac{\sin x}{\cos^2 x} = 0 \Rightarrow x = \frac{\pi}{6}$. Further,
$f''(x) = -\sec x(\tan^2 x + \sec^2 x) < 0$ on $(\frac{-\pi}{2}, \frac{\pi}{2})$. Therefore, f attains the maximum value $f(\frac{\pi}{6}) = \frac{\pi}{9} - \frac{2}{\sqrt{3}}$ on $\mathcal{I}$.

Global Extrema on Open Intervals (contd)

As another example, let us find the dimensions of the cone with minimum volume that can contain a sphere with radius R. Let h be the height of the cone and r the radius of its base. The objective to be minimized is the volume $f(r,h) = \frac{1}{3}\pi r^2 h$. The constraint betwen r and h is shown in Figure 11. The traingle AEF is similar to traingle ADB and therefore, $\frac{h-R}{R} = \frac{\sqrt{h^2+r^2}}{r}$.

Global Extrema on Open Intervals (contd)

Our first step is to reduce the volume formula to involve only one of r^2[3] or h.
The algebra involved will be the simplest if we solved for h.
The constraint gives us $r^2 = \frac{R^2 h}{h-2R}$. Substituting this expression for r^2 into the volume formula, we get $g(h) = \frac{\pi R^2}{3}\frac{h^2}{(h-2R)}$ with the domain given by $\mathcal{D} = \left\{h | 2R < h < \infty\right\}$.
Note that $\mathcal{D}$ is an open interval.
$g' = \frac{\pi R^2}{3}\frac{2h(h-2R)-h^2}{(h-2R)^2} = \frac{\pi R^2}{3}\frac{h(h-4R)}{(h-2R)^2}$ which is 0 in its domain $\mathcal{D}$ if and only if $h = 4R$.
$g'' = \frac{\pi R^2}{3}\frac{2(h-2R)^3-2h(h-4R)(h-2R)^2}{(h-2R)^4} = \frac{\pi R^2}{3}\frac{2(h^2-4Rh+4R^2-h^2+4Rh)}{(h-2R)^3} = \frac{\pi R^2}{3}\frac{8R^2}{(h-2R)^3}$, which is greater than 0 in $\mathcal{D}$.
Therefore, g (and consequently f) has a unique minimum at $h = 4R$ and correspondingly, $r^2 = \frac{R^2 h}{h-2R} = 2R^2$.

[3] Since r appears in the volume formula only in terms of r^2.

References

- Online Gradient Descent: Efficient Algorithm for Regret Minimization - Zinkevich 2005
- Yu-Hong Dai, Roger Fletcher. New algorithms for singly linearly constrained quadratic programs subject to lower and upper bounds. http://link.springer.com/content/pdf/10.1007%2Fs10107-005-0595-2.pdf
- Kalai-Vempala 2005
- Regret bound for Adagrad: Duchi, Hazan, Singer 2010