# Critical Point and Local Extreme Value

Given a critical point $c$, the following test helps determine if $f(c)$ is a local extreme value:

## Procedure

**[Local Extreme Value]:** *Let $c$ be an isolated critical point of $f$*

1. *$f(c)$ is a local minimum if $f(x)$ is decreasing in an interval $[c - \epsilon_1, c]$ and increasing in an interval $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ is increasing in an interval $[c - \epsilon_1, c]$ and decreasing in an interval $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*

Given a critical point $c$, *first derivative test* (sufficient condition) helps determine if $f(c)$ is a local extreme value:

## Procedure

**[First derivative test]:** *Let $c$ be an isolated critical point of $f$*

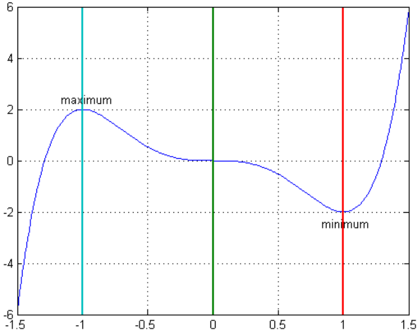
1. *$f(c)$ is a local minimum if the sign of $f'(x)$ changes from negative in $[c - \epsilon_1, c]$ to positive in $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
2. *$f(c)$ is a local maximum if $f(x)$ the sign of $f'(x)$ changes from positive in $[c - \epsilon_1, c]$ to negative in $[c, c + \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$.*
3. *If $f'(x)$ is positive in an interval $[c - \epsilon_1, c]$ and also positive in an interval $[c, c - \epsilon_2]$, or $f'(x)$ is negative in an interval $[c - \epsilon_1, c]$ and also negative in an interval $[c, c - \epsilon_2]$ with $\epsilon_1, \epsilon_2 > 0$, then $f(c)$ is not a local extremum.*

# First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has

# First Derivative Test: Critical Point and Local Extreme Value

As an example, the function $f(x) = 3x^5 - 5x^3$ has the derivative $f'(x) = 15x^2(x+1)(x-1)$. The critical points are $0$, $1$ and $-1$. Of the three, the sign of $f'(x)$ changes at $1$ and $-1$, which are local minimum and maximum respectively. The sign does not change at $0$, which is therefore not a local supremum.



# First Derivative Test: Critical Point and Local Extreme Value

As another example, consider the function

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then,

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Note that $f(x)$ is discontinuous at $x = 0$, and therefore $f'(x)$ is not defined at $x = 0$. All numbers $x \geq 0$ are critical numbers. $f(0) = 0$ is a local minimum, whereas $f(x) = 1$ is a local minimum as well as a local maximum $\forall x > 0$.

# Strict Convexity and Extremum

- A differentiable function $f$ is said to be *strictly convex* (or *strictly concave up*) on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is increasing on $\mathcal{I}$.
- Recall from theorem 7, the graphical interpretation of the first derivative $f'(x)$; $f'(x) > 0$ implies that $f(x)$ is increasing at $x$.
- Similarly, $f'(x)$ is increasing when $f''(x) > 0$. This gives us a sufficient condition for the strict convexity of a function:

## Claim

*If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) > 0$, $\forall x \in \mathcal{I}$, then the slope of the function is always increasing with $x$ and the graph is strictly convex. This is illustrated in Figure 8.*

- On the other hand, if the function is strictly convex and doubly differentiable in $\mathcal{I}$, then $f''(x) \geq 0$, $\forall x \in \mathcal{I}$.

# Strict Convexity and Extremum (Illustrated)

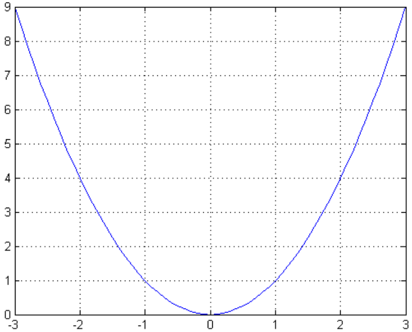



Figure 8:

# Strict Convexity and Extremum: Slopeless interpretation (SI)

## Claim

*A function $f$ is strictly convex on an open interval $\mathcal{I}$, iff*

$$f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2) \quad (1)$$

*whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.*

# SI: Necessity when $f$ is differentiable

First we will prove the **necessity**.

[2] For the case $x_2 < x_1$, the proof is very similar.

# SI: Necessity when $f$ is differentiable

First we will prove the **necessity**. Suppose $f'$ is increasing on $\mathcal{I}$. Let $0 < a < 1$, $x_1, x_2 \in \mathcal{I}$ and $x_1 \neq x_2$. Without loss of generality assume that[2] $x_1 < x_2$. Then, $x_1 < ax_1 + (1-a)x_2 < x_2$ and therefore $ax_1 + (1-a)x_2 \in \mathcal{I}$. By the mean value theorem, there exist $s$ and $t$ with $x_1 < s < ax_1 + (1-a)x_2 < t < x_2$, such that $f(ax_1 + (1-a)x_2) - f(x_1) = f'(s)(x_2 - x_1)(1-a)$ and $f(x_2) - f(ax_1 + (1-a)x_2) = f'(t)(x_2 - x_1)a$. Therefore,

[2] For the case $x_2 < x_1$, the proof is very similar.

# SI: Necessity when $f$ is differentiable

First we will prove the **necessity**. Suppose $f'$ is increasing on $\mathcal{I}$. Let $0 < a < 1$, $x_1, x_2 \in \mathcal{I}$ and $x_1 \neq x_2$. Without loss of generality assume that[2] $x_1 < x_2$. Then, $x_1 < ax_1 + (1-a)x_2 < x_2$ and therefore $ax_1 + (1-a)x_2 \in \mathcal{I}$. By the mean value theorem, there exist $s$ and $t$ with $x_1 < s < ax_1 + (1-a)x_2 < t < x_2$, such that $f(ax_1 + (1-a)x_2) - f(x_1) = f'(s)(x_2 - x_1)(1-a)$ and $f(x_2) - f(ax_1 + (1-a)x_2) = f'(t)(x_2 - x_1)a$. Therefore,

$$\begin{aligned} (1-a)f(x_1) - f(ax_1 + (1-a)x_2) + af(x_2) &= \\ a\left[f(x_2) - f(ax_1 + (1-a)x_2)\right] - (1-a)\left[f(ax_1 + (1-a)x_2) - f(x_1)\right] &= \\ a(1-a)(x_2 - x_1)\left[f'(t) - f'(s)\right] & \end{aligned}$$

Since $f(x)$ is strictly convex on $\mathcal{I}$, $f'(x)$ is increasing on $\mathcal{I}$ and therefore, $f'(t) - f'(s) > 0$. Moreover, $x_2 - x_1 > 0$ and $0 < a < 1$. Thus, $(1-a)f(x_1) - f(ax_1 + (1-a)x_2) + af(x_2) > 0$, or equivalently, $f(ax_1 + (1-a)x_2) < af(x_1) + (1-a)f(x_2)$, which is what we wanted to prove in inequality (1).

[2] For the case $x_2 < x_1$, the proof is very similar.

# SI: Sufficiency when $f$ is differentiable

Suppose the inequality in inqeuality (1) holds. Then, taking limits,

# SI: Sufficiency when $f$ is differentiable

Suppose the inequality in inqeuality (1) holds. Then, taking limits, $\lim_{a \to 0} \frac{f(x_2 + a(x_1 - x_2)) - f(x_2)}{a} \leq f(x_1) - f(x_2)$. That is,

# SI: Sufficiency when $f$ is differentiable

Suppose the inequality in inqeuality (1) holds. Then, taking limits, $\lim_{a \to 0} \frac{f(x_2 + a(x_1 - x_2)) - f(x_2)}{a} \leq f(x_1) - f(x_2)$. That is,

$$f'(x_2)(x_1 - x_2) \leq f(x_1) - f(x_2) \tag{2}$$

Similarly, we can show that

$$f'(x_1)(x_2 - x_1) \leq f(x_2) - f(x_1) \tag{3}$$

Adding the left and right hand sides of inequalities in (2) and (3), and multiplying the resultant inequality by $-1$ gives us

# SI: Sufficiency when $f$ is differentiable

Suppose the inequality in inqeuality (1) holds. Then, taking limits, $\lim_{a \to 0} \frac{f(x_2 + a(x_1 - x_2)) - f(x_2)}{a} \leq f(x_1) - f(x_2)$. That is,

$$f'(x_2)(x_1 - x_2) \leq f(x_1) - f(x_2) \tag{2}$$

Similarly, we can show that

$$f'(x_1)(x_2 - x_1) \leq f(x_2) - f(x_1) \tag{3}$$

Adding the left and right hand sides of inequalities in (2) and (3), and multiplying the resultant inequality by $-1$ gives us

$$\left(f'(x_2) - f'(x_1)\right)(x_2 - x_1) \geq 0 \tag{4}$$

We now need to prove that the inequality in (4) is strict.

# SI: Sufficiency when $f$ is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

# SI: Sufficiency when $f$ is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

$$f(x_2) - f(x_1) = f'(z)(x_2 - x_1) \tag{5}$$

Since (4) holds for any $x_1, x_2 \in \mathcal{I}$, it also hold for $x_2 = z$. Therefore,

# SI: Sufficiency when $f$ is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

$$f(x_2) - f(x_1) = f'(z)(x_2 - x_1) \tag{5}$$

Since (4) holds for any $x_1, x_2 \in \mathcal{I}$, it also hold for $x_2 = z$. Therefore,

$$(f'(z) - f'(x_1))(x_2 - x_1) = \frac{1}{t}(f'(z) - f'(x_1))(z - x_1) \geq 0$$

Additionally using (5), we get

# SI: Sufficiency when $f$ is differentiable (contd)

Using the mean value theorem, $\exists z = x_1 + t(x_2 - x_1)$ for $t \in (0, 1)$ such that

$$f(x_2) - f(x_1) = f'(z)(x_2 - x_1) \tag{5}$$

Since (4) holds for any $x_1, x_2 \in \mathcal{I}$, it also hold for $x_2 = z$. Therefore,

$$(f'(z) - f'(x_1))(x_2 - x_1) = \frac{1}{t}(f'(z) - f'(x_1))(z - x_1) \geq 0$$

Additionally using (5), we get

$$f(x_2) - f(x_1) = (f'(z) - f'(x_1))(x_2 - x_1) + f'(x_1)(x_2 - x_1) \geq f'(x_1)(x_2 - x_1) \tag{6}$$

# SI: Sufficiency when $f$ is differentiable (contd)

Suppose equality holds in (4) for some $x_1 \neq x_2$. Then it holds in (6) for the same $x_1$ and $x_2$. That is,

# SI: Sufficiency when $f$ is differentiable (contd)

Suppose equality holds in (4) for some $x_1 \neq x_2$. Then it holds in (6) for the same $x_1$ and $x_2$. That is,

$$f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1) \tag{7}$$

Substituting $x_2$ with $x_1 + a(x_2 - x_1)$ and applying (6), we get

# SI: Sufficiency when $f$ is differentiable (contd)

Suppose equality holds in (4) for some $x_1 \neq x_2$. Then it holds in (6) for the same $x_1$ and $x_2$. That is,

$$f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1) \tag{7}$$

Substituting $x_2$ with $x_1 + a(x_2 - x_1)$ and applying (6), we get

$$f(x_1) + af'(x_1)(x_2 - x_1) \leq f(x_1 + a(x_2 - x_1)) \tag{8}$$

Further using (1) and 7, we can derive that

# SI: Sufficiency when $f$ is differentiable (contd)

Suppose equality holds in (4) for some $x_1 \neq x_2$. Then it holds in (6) for the same $x_1$ and $x_2$. That is,

$$f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1) \tag{7}$$

Substituting $x_2$ with $x_1 + a(x_2 - x_1)$ and applying (6), we get

$$f(x_1) + af'(x_1)(x_2 - x_1) \leq f(x_1 + a(x_2 - x_1)) \tag{8}$$

Further using (1) and 7, we can derive that

$$f(x_1 + a(x_2 - x_1)) < (1 - a)f(x_1) + af(x_2) = f(x_1) + af'(x_1)(x_2 - x_1) \tag{9}$$

Contradiction between (8) and (9)!

# SI: Sufficiency when $f$ is differentiable (contd)

Thus, equations 8 and 9 contradict each other.

$$f(x_1) + af'(x_1)(x_2 - x_1) \leq f(x_1 + a(x_2 - x_1))$$

$$f(x_1 + a(x_2 - x_1)) < (1 - a)f(x_1) + af(x_2) = f(x_1) + af'(x_1)(x_2 - x_1)$$

Therefore, equality in 4 cannot hold for any $x_1 \neq x_2$, implying that

$$\left(f'(x_2) - f'(x_1)\right)(x_2 - x_1) > 0$$

that is, $f'(x)$ is increasing and therefore $f$ is convex on $\mathcal{I}$. □

# Strict Concavity

- A differentiable function $f$ is said to be *strictly concave* on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is decreasing on $\mathcal{I}$.
- Recall from theorem 7, the graphical interpretation of the first derivative $f'(x)$; $f'(x) < 0$ implies that $f(x)$ is decreasing at $x$.
- Similarly, $f'(x)$ is (strictly) monotonically decreasing when

# Strict Concavity

- A differentiable function $f$ is said to be *strictly concave* on an open interval $\mathcal{I}$, *iff*, $f'(x)$ is decreasing on $\mathcal{I}$.
- Recall from theorem 7, the graphical interpretation of the first derivative $f'(x)$; $f'(x) < 0$ implies that $f(x)$ is decreasing at $x$.
- Similarly, $f'(x)$ is (strictly) monotonically decreasing when $f''(x) > 0$. This gives us a sufficient condition for the concavity of a function:

**Claim**

*If at all points in an open interval $\mathcal{I}$, $f(x)$ is doubly differentiable and if $f''(x) < 0, \ \forall x \in \mathcal{I}$, then the slope of the function is always decreasing with $x$ and the graph is strictly concave.*

# Strict Concavity

On the other hand, if the function is strictly concave and doubly differentiable in $\mathcal{I}$, then $f''(x) \leq 0,\ \forall x \in \mathcal{I}$. This is illustrated in Figure 9.

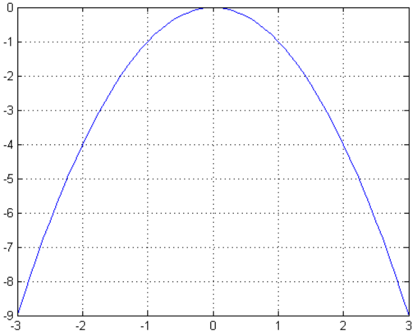



Figure 9:

# Strict Concavity (slopeless interpretation)

There is also a slopeless interpretation of concavity as stated in the following theorem:

## Claim

*A differentiable function f is strictly concave on an open interval $\mathcal{I}$, iff*

$$f(ax_1 + (1-a)x_2) > af(x_1) + (1-a)f(x_2) \tag{10}$$

*whenver $x_1, x_2 \in \mathcal{I}$, $x_1 \neq x_2$ and $0 < a < 1$.*

The proof is similar to that for theorem 12.

# Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$.

# Convex & Concave Regions and Inflection Point

Study the function $f(x) = x^3 - x + 2$. It's slope decreases as $x$ increases to $0$ ($f'(x) < 0$) and then the slope increases beyond $x = 0$ ($f'(x) > 0$). The point $0$, where the $f'(x)$ changes sign is called the *inflection point*; the graph is strictly concave for $x < 0$ and strictly convex for $x > 0$. See Figure 10.