Euclidean balls and ellipsoids

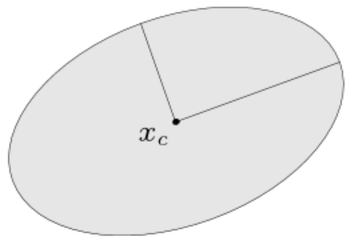
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \ge 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \le \|x\| + \|y\|$

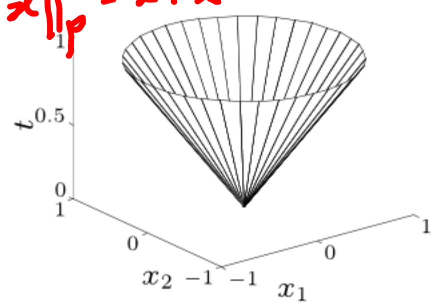
notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \le r\}$

Eucledian ball $\Rightarrow \| \; \|_2$ Ellipsoid $\Rightarrow \|x\|_P^2 = x^T P x$

norm cone: $\{(x, t) \mid \|x\| \le t\}$

Euclidean norm cone is called second-order cone



norm balls and cones are convex

Prove that under specific assumptions on P, $\sqrt{x^T P x}$ is a valid norm. Assume $x \in \mathbb{R}^n$ & $P \in \mathbb{R}^{n \times n}$

Proof: Suppose P is symmetric positive definite:

i.e $P^T = P$ & $\forall x \neq 0 \quad x^T P x > 0$

① $\|x\|_p^2 = x^T P x \geq 0$ with equality iff $x = 0$ (obvious)

② $\|\alpha x\|_p = \sqrt{\alpha^2 x^T P x} = |\alpha| \sqrt{x^T P x} = |\alpha| \, \|x\|_p$

③ $\|x+y\|_p^2 = (x+y)^T P (x+y) = x^T P x + y^T P y + 2 x^T P y$

(since $P = P^T \Rightarrow x^T P y = y^T P x = (x^T P y)^T$)

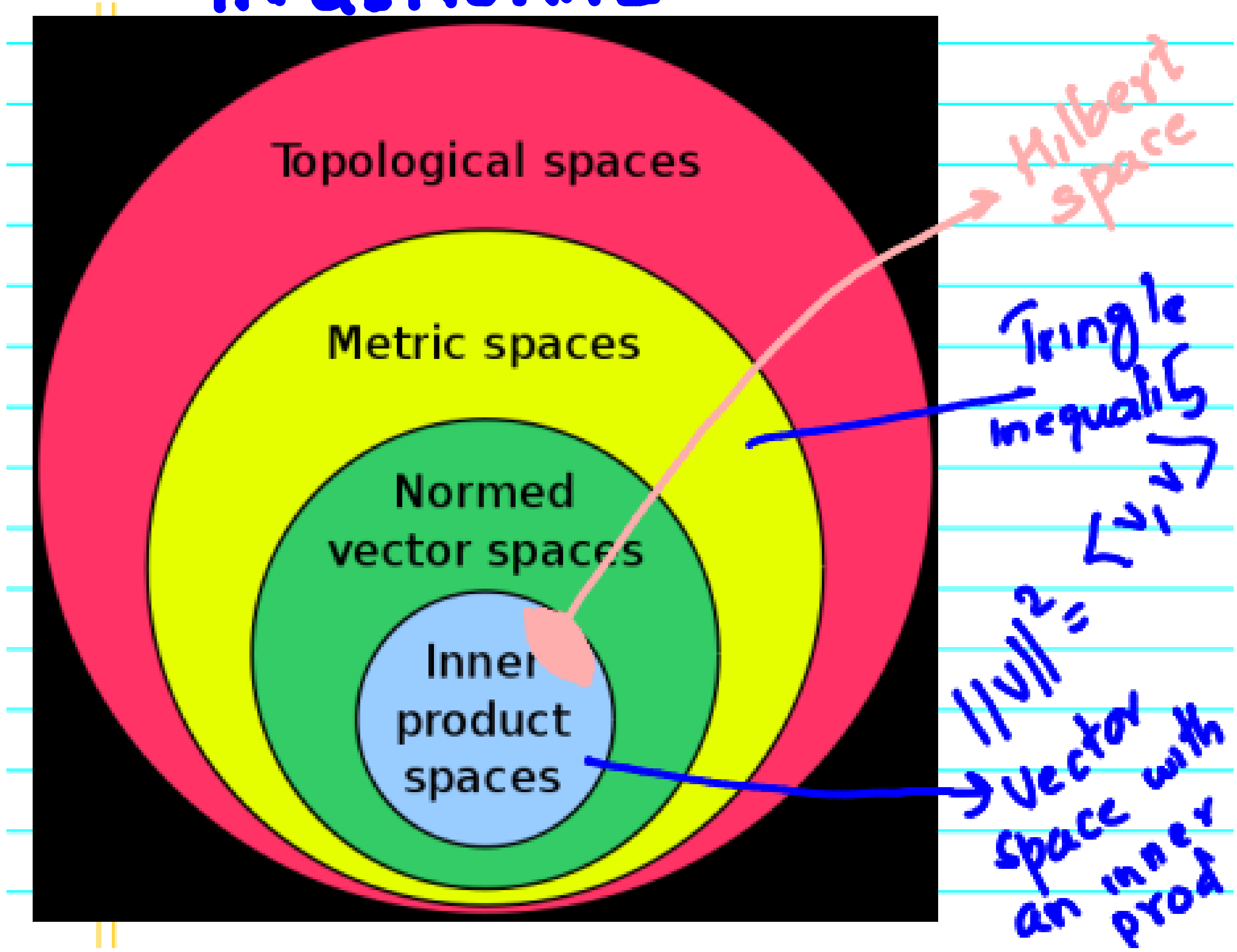
$$= \|x\|_p^2 + \|y\|_p^2 + 2 x^T P y$$

(need to prove) $\leq \|x\|_p^2 + \|y\|_p^2 + 2\|x\|_p \|y\|_p$

Using positive definiteness of P, can you prove that

$$x^T P y \leq \sqrt{x^T P x} \sqrt{y^T P y} \ ?$$

IN GENERAL



Source: http://en.wikipedia.org/wiki/Space_(mathematics)

A hierarchy of mathematical spaces: The inner product induces a norm. The norm induces a metric. The metric induces a topology.

Topological space: Set of points along with a set of neighborhoods of each point, with certain axioms required to be satisfied by the pts & their nbrhoods

Metric space: Set of points with a notion of "distance" between elements
$d(x,y)$ must be (a) non-negative (b) $d(x,y)=0$ iff $x=y$ (c) symmetric (d) satisfy triangle inequality

Assuming you have understood vector space

Normed vector space: A vector space on which a norm is defined. (see previous page for definition of norm)

[H/w: Prove that "normed" space is a "metric" space]

Inner product space: It is a vector space over a field of scalars along with an inner product

eg: $\mathbb{R}$

an algebraic structure with addition, subtraction, multiplication & division

associative & commutative

must be commutative associative distributive

multiplicative inverse must exist

ⓐ (Conjugate) symmetry:

$$\langle x, y\rangle = \langle y, x\rangle$$

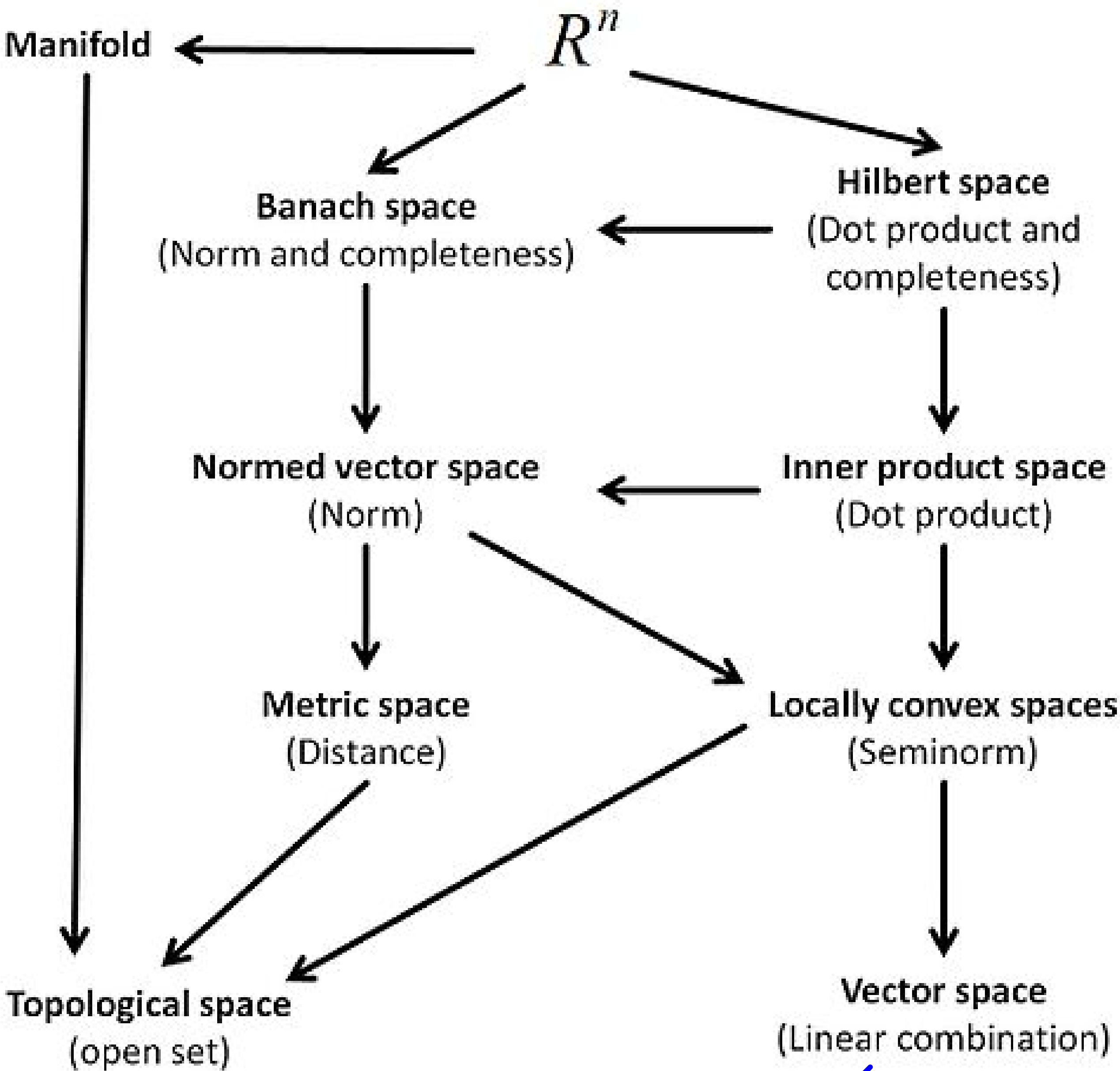
ⓑ Linearity in the first argument

$$\langle ax, y\rangle = a\langle x, y\rangle$$

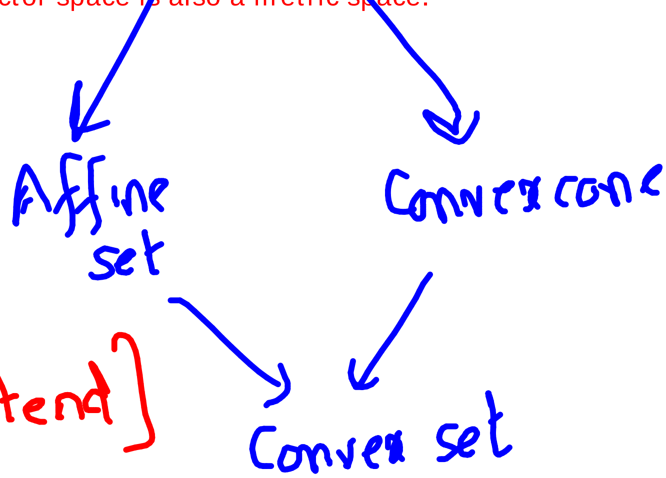
$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

ⓒ Positive definiteness:

$\langle x, x\rangle \geq 0$ with equality iff $x = 0$



Overview of types of abstract spaces. An arrow from space A to space B implies that space A is also a kind of space B. That means, for instance, that a normed vector space is also a metric space.



[H/W: Confirm or contend]