Recap: We wanted to generalise vector spaces to beyond $\mathbb{R}^n$ [& therefore norms & inner prods & basis & dimensions etc]

## Euclidean balls and ellipsoids

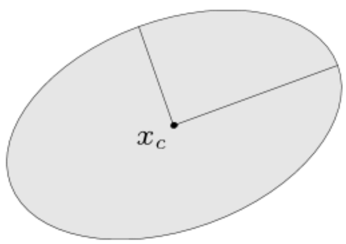
**(Euclidean) ball** with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

**ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, $P$ symmetric positive definite)




other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with $A$ square and nonsingular



## Norm balls and norm cones

**norm:** a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

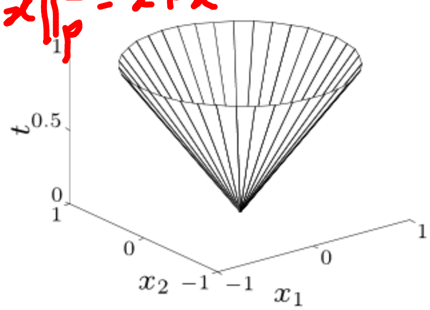
notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

**norm ball** with center $x_c$ and radius $r$: $\{x \mid \|x - x_c\| \leq r\}$

Euclidean ball $\Rightarrow \| \; \|_2$    Ellipsoid $\Rightarrow \|x\|_P^2 = x^T P x$

**norm cone:** $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone




norm balls and cones are convex

- 31/07/2013. Show that the following are vector spaces (assuming scalars come from a set S), and then answer questions that follow for each of them: Set of all matrices on S, set of all polynomials on S, set of all sequences of elements of S. (HINT: You can refer to this book for answers to most questions in this homework.) How would you understand the concepts of independence, span, basis, dimension and null space (chapter 2 of this book), eigenvalues and eigenvectors (chapter 5), inner product and orthogonality (chapter 6)? EXTRA: Now how about set of all random variables and set of all functions. **Deadline:** August 7 2013.

1) For understanding concepts of eigenvalues & eigenvectors, you need concept of linear operator : $T : v \to v'$   See ch 5 of

http://athena.nitc.ac.in/~kmurali/dms/axler.pdf

$v \in V$    $v' \in V$

Eg for polynomials & fn spaces, T could be $\frac{d}{dx}$. See Eigenfunctions : http://en.wikipedia.org/wiki/Eigenfunction

2) For basis for polynomials/fns, see basis function:

http://en.wikipedia.org/wiki/Basis_function

3) For the concept of null space, see Kernel:

https://en.wikipedia.org/wiki/Kernel_(linear_algebra)

4) As for inner product:

For $L_2$ functions: $\langle f, g \rangle = \int_{Dmn} f(x) g(x) dx$

i.e functions f s.t $\|f\|_2 = \left(\int_{Dmn} |f|^2 dx\right)^{1/2} < \infty$

$\langle f, g \rangle = \int_{Dmn} \int_{Dmn} f(x) g(y) w(x,y) \, dx \, dy$

$w(x,y)$ should be a positive def. fn

## Compact representation of a vector space:

Let $S \subset V$ be a linear set with an inner product $\langle , \rangle$

linear subspace

Let $B = \text{basis}(S)$

Let $\dim(V) = n$ & $\dim(S) = m \le n$

Define $\boxed{S^\perp = \{v \in V \mid \langle v,u\rangle = 0 \ \forall u \in S\}}$

$S^\perp$ is the orthogonal complement of $S$

prove that $S^\perp$ is a vector space

→ $S^\perp$ & $S$ both are linear subspaces of $V$

→ Let $B^\perp$ be a basis for $S^\perp$

by Rank nullity thm

→ Then $S^\perp \cap S = \boxed{\{0\}}$, $\dim(S) + \dim(S^\perp) = \boxed{n}$

→ A basis for $V$ is $\boxed{B \cup B^\perp}$, $(S^\perp)^\perp = \boxed{S}$

→ And $\{v \in V \mid \langle v,u\rangle = 0 \ \forall u \in B^\perp\} = \boxed{S}$

$\{v \in V \mid \langle v,u\rangle = 0 \ \forall u \in B\} = \boxed{S^\perp}$

[Ref: Appendix A of http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf ]

# Dual representation: Explained with analogy

$$\underbrace{\begin{bmatrix} 0 & & 0 \\ & \ddots 1 \ddots & \\ 0 & & 0 \end{bmatrix}}_{n} \cdot \underbrace{\begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 0 & 1 & 0 \end{bmatrix}}_{\frac{n(n-1)}{2}} \qquad \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ & \ddots & \\ 0 & -1 & 0 \end{bmatrix}}_{\frac{n(n-1)}{2}}$$

Q: What is $B$ & $B^{\perp}$ if $S$ = space of symmetric matrices? (assume Frobenius i.p.)

$B^{\perp}$ has smaller size than $B$

$B$ = compact representation ~~for~~ $S$ when: $m < n-m$

$B^{\perp}$ = compact representation for $S$ when: $m > n-m$

Eg: $S \subseteq \mathbb{R}^n$ & $a_1, a_2 \ldots a_n$ is a finite spanning set in $S^{\perp}$

$$\Rightarrow S = (S^{\perp})^{\perp} = \{x \mid a_i^T x = 0;\ i = 1 \ldots k\}$$

A dual representation for $S$ $\leftarrow$ $= \boxed{\{x \mid Ax = 0;\ a_i^T \text{ is } i^{th} \text{ row of } A\}}$

Dual representation of linear subspace in $\mathbb{R}^n$

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Now recall affine sets: (say $A \subseteq \mathbb{R}^n$)

(a) $A$ is affine iff $\forall\, u, v \in A$ $\theta u + (1-\theta) v \in A$ $\forall\, \theta \in \mathbb{R}$

$\Updownarrow$ iff

V shifted by u

(b) A is affine iff $A = \{u + v \mid u \in \mathbb{R}^n \text{ is fixed } \& \ v \in V\}$

for some vector space $V \subseteq \mathbb{R}^n$

$\Updownarrow$ iff

(c) A is affine iff $A = \{x \mid Px = b\}$

for some P (whose rank $= \boxed{n - \dim(V)}$)

and b

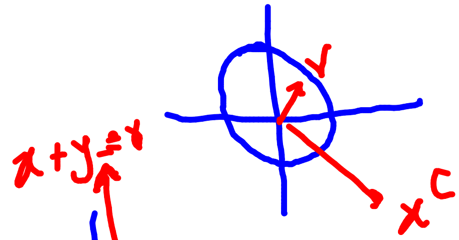
Thus, hyperplanes are affine spaces of dimension $\boxed{n-1}$ with $Px = b$ given by $p^T x = b$

We will soon see duality for convex cones & in general, convex sets

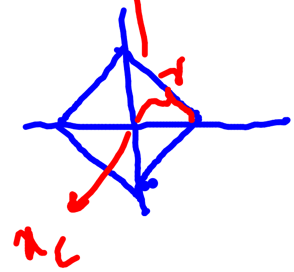
[ASIDE]

What do norm balls (say in $\mathbb{R}^2$) correspond to?

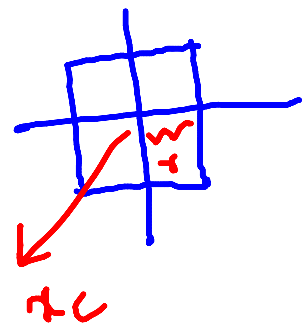
① $\|x - x_c\|_2 \leq r \Rightarrow$




② $\|x - x_c\|_1 \leq r \Rightarrow$




③ $\|x - x_c\|_\infty \leq r \Rightarrow$




$\| \ \|_1$ is often used in optimisation problems since soln with $\| \ \|_1 = k$ is probably going to have lots of zero components: SPARSITY INDUCING NORM

$\| \ \|_2$ is not

$\| \ \|_1$ is

$\| \ \|_\infty$ is

$\| \ \|_p$

$\| \ \|_{p \in [1,2]}$ $0 < p < 1$