Normed Vector Spaces

- **Vector Space:** A space consisting of vectors, together with the
  1. **associative and commutative operation of addition of vectors**,
  2. **associative and distributive operation of multiplication of vectors by scalars.**
- **Norm:** A function that assigns a strictly positive length or size to each vector in a vector space — save for the zero vector, which is assigned a length of zero.
- **Normed Vector Space:** A vector space on which a norm is defined.

Normed Vector Spaces

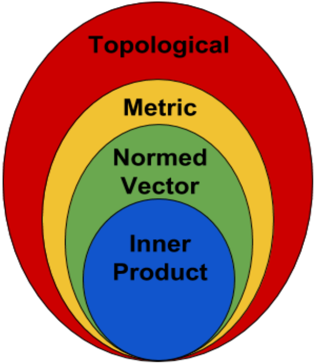
A vector space on which a norm is defined.

- In any real vector space $\Re^n$, the length of a vector has the following properties:
  1. The zero vector, 0, has zero length; every other vector has a positive length.
     - $||\mathbf{x}|| \geq 0$, and $||\mathbf{x}|| = 0$ iff $\mathbf{x} = 0$.
  2. Multiplying a vector by a positive number changes its length without changing its direction. Moreover,
     - $||\alpha \mathbf{x}|| = |\alpha| ||\mathbf{x}||$ for any scalar α.
  3. The triangle inequality holds. That is, taking norms as distances, the distance from point A through B to C is never shorter than going directly from A to C, or the shortest distance between any two points is a straight line.
     - $||\mathbf{x}_1 + \mathbf{x}_2|| \leq ||\mathbf{x}_1|| + ||\mathbf{x}_2||$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

The generalization of these three properties to more abstract vector spaces leads to the notion of norm. For example: A matrix norm.
Additionally, in the case of square matrices (thus, m = n), some (but not all) matrix norms satisfy the following condition, which is related to the fact that matrices are more than just vectors: $||\mathbf{AB}|| \leq ||\mathbf{A}||\ ||\mathbf{B}||$ **for all matrices A and B in** K^{nxn}**.**

Contrasting the Spaces discussed so far

- Topological Spaces: Notion of neighbourhood of points.
- Metric Spaces: Notion of positive distance between two points.
- Normed Vector Spaces: Notion of positive length of each point.



Topological Spaces

Set of points X along with the set of open sets ($\mathcal{N}$) with certain axioms required to be satisfied by sets in $\mathcal{N}$:

- Definition 1: A topological space is an ordered pair (X,$\mathcal{N}$), where:
  - X is a set
  - $\mathcal{N}$ is a collection of subsets of X, satisfying the following axioms:
    - ★ The empty set and X itself belong to $\mathcal{N}$.
    - ★ Any (finite or infinite) union of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.
    - ★ The intersection of any finite number of members of $\mathcal{N}$ still belongs to $\mathcal{N}$.
- We already saw examples that are (and are not) toplogies for $X = \{1,2,3\}$ and $\mathcal{N} =$
  - $\{\{\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$ **Yes**
  - $\{\{\},\{1\},\{2\},\{1,2,3\}\}$ **No as {1}** $\cup$ **{2}** $\notin \mathcal{N}$
  - $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$ **No as {1,2}** $\cap$ **{2,3}** $\notin \mathcal{N}$

Topological Spaces and Open Sets

The neighbourhoods can be recovered by defining $N(\mathbf{x})$ to be a neighbourhood of $\mathbf{x}$ if $\mathcal{N}$ includes a set O such that $\mathbf{x} \in O$. The sets $O \in \mathcal{N}$ are basically the open sets. For example

- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\}, \{1,2,3\}$, each of $\{\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1,2,3\}\}$
  $N(2) \in \{\{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$
- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\}, \{1\}, \{1,2,3\}\}$, each of $\{\}$,$\{1\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1\}, \{1,2,3\}\}$
  $N(2) \in \{\{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$

  Expect N(.) to contain intersections, unions, supers of its elements
- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$, each of $\{\}, \{1\}, \{2\}, \{1,2\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1\}, \{1,2\}, \{1,2,3\}\}$
  $N(2) \in \{\{2\}, \{1,2\}, \{1,2,3\}\}$
  $N(3) \in \{\{1,2,3\}\}$

Topological Spaces and Open Sets

- with $X = \{1,2,3\}$ and $\mathcal{N} = \{\{\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ each of $\{\}, \{2\}, \{1,2\}, \{2,3\}$ and $\{1,2,3\}$ is an open set O and
  $N(1) \in \{\{1,2\}, \{1,2,3\}\}$
  $N(2) \in \{\{1,2\}, \{2,3\}, \{1,2,3\}\}$,{2}
  $N(3) \in \{\{2,3\}, \{1,2,3\}\}$

May not make too much sense for X=finite collection of points

(Alternative) Definition 2: A topological space is an ordered pair $(X, N(.))$, where X is a set and $N(.)$ is a neighborhood function such that for each $\mathbf{x} \in X$, if $N(\mathbf{x})$ is a

- neighbourhood of $\mathbf{x}$ then $\mathbf{x} \in N(\mathbf{x})$.
- subset of X and includes a neighbourhood of $\mathbf{x}$, then $N(bfx)$ is a neighbourhood of $\mathbf{x}$.
- neighbourhood of $\mathbf{x}$, then for any other neighborhood $N'(\mathbf{x})$, $N(\mathbf{x}) \cap N'(\mathbf{x})$ is also a neighbourhood of $\mathbf{x}$.
- neighbourhood of $\mathbf{x}$, then it includes a neighbourhood $N'(\mathbf{x})$ such that $N(\mathbf{x})$ is a neighbourhood of each point of $N'(\mathbf{x})$.

The underlying motivation for calling these sets as open sets

What topological spaces (and their special cases) give us

- Definition 1: A topological space is an ordered pair $(X,\mathcal{N})$, where..... An enumeration of open sets
- Definition 2: A topological space is an ordered pair $(X,N(.))$, where..... A neighborhood function
- Definition 1 allows for understanding open sets as elements of $\mathcal{N}$.
  - We can define an open ball $B(\mathbf{x})$ to be any element of $N(\mathbf{x})$.
  - If additionally, we have metric $d(.,.)$ on the space, we can define an open ball $B(\mathbf{x},r)$ of radius r as $\{\mathbf{y}|\mathbf{d}(\mathbf{x},\mathbf{y})<\mathbf{r}\}$
  - A norm ball $B(\mathbf{x},r)=\{\mathbf{y}|\|\mathbf{x}-\mathbf{y}\|<\mathbf{r}\}$ also should have homogenity! That is, $\|\alpha\mathbf{x}-\alpha\mathbf{y}\|=|\alpha|\|\mathbf{x}-\mathbf{y}\|$ With norms, you can talk of Canonical balls!
- Definition 2 allows for continuity of function f definied from a topology $X,N(.)$ to another topology $Y,M(.)$. Function f is continuous if for every $\mathbf{x}\in X$ and every neighbourhood $M(f(\mathbf{x}))$ of $f(\mathbf{x})$ there is a neighbourhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $f(N(\mathbf{x}))\subseteq$ ~~$M(\mathbf{x})$~~ M(f(x))

Example of a canonical ball is norm ball of radius 1

HW1: A Topological space that does not have metric

Consider $X = \{0, 1\}$ and $\mathcal{N} = \{\emptyset, \{0\}, \{0, 1\}\}$,
Consider some metric $d(.,.)$ which is 0 if both its arguments are the same and 1 otherwise. If d would be such a metric, a neighborhood (ball) of radius 0.5 around 1, that is $B(1, 0.5)$ would equal $\{1\}$, which should have been open. However, $\{1\} \notin \mathcal{N}$. Contradiction!

HW2: A metric space that does not have norm

Consider (again) the **discrete** metric $d(.,.)$ over a vector space V. We define $d(.,.)$ to be 0 if both its arguments are the same and 1 otherwise. While one can verify that this metric satisifies the triangle inequality, what one requires from an equivalent norm $\|.\|_n$ is that for any $\mathbf{x}, \mathbf{y} \in V$, with $\mathbf{x} \neq \mathbf{y}$, for any scalar $\alpha \neq 0$, we must have $\|\alpha\mathbf{x} - \alpha\mathbf{y}\|_n = \alpha\|\mathbf{x} - \mathbf{y}\|_n$. This measure using the norm can clearly not correspond to the **discrete** distance metric.

Inner Product Space

It is a vector space over a field of scalars along with an inner product.

- **Field of scalars:** e.g. IR algebraic structure with:-
  1. Addition: must be multiplicative and associative.
  2. Subtraction.
  3. Multiplication: must be commutative, associative and distributive.
  4. Division: multiplicative inverse must exist.
- **Inner Product:**
  1. (Conjugate) Symmetry: $<\mathbf{x}_1, \mathbf{x}_2> = \overline{<\mathbf{x}_2, \mathbf{x}_1>}$. Conjugacy is when scalars are allowed to be complex
  2. Linearity in the first argument.
     - $< a\mathbf{x}_1, \mathbf{x}_2 >= a < \mathbf{x}_1, \mathbf{x}_2 >$
     - $< \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_3 >=< \mathbf{x}_1, \mathbf{x}_3 > + < \mathbf{x}_3, \mathbf{x}_3 >$ Equality on projection to x3, Not otherwise..
  3. Positive definiteness: $< \mathbf{x}, \mathbf{x} >\geq 0$, with equality iff $\mathbf{x} = 0$.

(Recall triangle inequality for normed vector spaces)

Proof: Normed Vector Space is a Metric Space

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Metric Space: Set of points (X) along with a notion of distance $d(\mathbf{x}_1, \mathbf{x}_2)$ between any two points $(\mathbf{x}_1, \mathbf{x}_2 \in X)$ such that:
   1. $d(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ (non-negativity).
   2. $d(\mathbf{x}_1, \mathbf{x}_2) = 0$ iff $\mathbf{x}_1 = \mathbf{x}_2$ (identity).
   3. $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{x}_2, \mathbf{x}_1)$ (symmetry).
   4. $d(\mathbf{x}_1, \mathbf{x}_2) + d(\mathbf{x}_2, \mathbf{x}_3) \geq d(\mathbf{x}_1, \mathbf{x}_3)$ (triangle inequality).
3. Proof:

   Straightforward!

Proof: Normed Vector Space is a Metric Space

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Metric Space: Set of points (X) along with a notion of distance $d(\mathbf{x}_1, \mathbf{x}_2)$ between any two points $(\mathbf{x}_1, \mathbf{x}_2 \in X)$ such that:
   1. $d(\mathbf{x}_1, \mathbf{x}_2) \geq 0$ (non-negativity).
   2. $d(\mathbf{x}_1, \mathbf{x}_2) = 0$ iff $\mathbf{x}_1 = \mathbf{x}_2$ (identity).
   3. $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{x}_2, \mathbf{x}_1)$ (symmetry).
   4. $d(\mathbf{x}_1, \mathbf{x}_2) + d(\mathbf{x}_2, \mathbf{x}_3) \geq d(\mathbf{x}_1, \mathbf{x}_3)$ (triangle inequality).
3. Proof:
   1. In vector space, a vector $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ can be defined by subtraction. Define $d(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|$, so 1.1 $\Rightarrow \|\mathbf{x}_1 - \mathbf{x}_2\| \geq 0$; $\|\mathbf{x}_1 - \mathbf{x}_2\| = 0$ iff $\mathbf{x}_1 - \mathbf{x}_2 = 0$, hence 2.1 and 2.2 are proved.
   2. 1.2 $\Rightarrow \|-1(\mathbf{x}_1 - \mathbf{x}_2)\| = |-1|\|\mathbf{x}_1 - \mathbf{x}_2\|$. So, $\|\mathbf{x}_2 - \mathbf{x}_1\| = \|\mathbf{x}_1 - \mathbf{x}_2\|$, so 2.3 is proved.
   3. Take $\mathbf{x}_1 = \mathbf{z}_1 - z_0$ and $\mathbf{x}_2 = \mathbf{z}_0 - \mathbf{z}_2$, put in 1.3 to get $\|\mathbf{z}_1 - \mathbf{z}_0\| + \|\mathbf{z}_0 - \mathbf{z}_2\| \geq \|\mathbf{z}_1 - \mathbf{z}_2\|$ so 2.4 is prooved.

The Mathematical Structures & Spaces: Some Proofs

Some Proofs For Mathematical Structures & Spaces

- Under what conditions on P, is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?
- Prove that inner product space is a normed vector space. Cauchy Shwarz can be us
- What is an example of normed vector space that is not an inner product space?
- Prove that $| < u, v > | \leq \|u\|_P \|v\|_P$ for any norm P. Cauchy Shwarz...

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Assume $\mathbf{x} \in \Re^n$ and $P \in \Re^{n \times n}$.

1. P is symmetric positive definite iff:
   1. Symmetric: $P^T = P$
   2. Positive Definite: $\forall \mathbf{x} \neq 0,\ \mathbf{x}^T P \mathbf{x} \geq 0$

Proof:

All eigenvalues of P are non-negative
Orthonormal basis for column space of P using eigenvectors
Express x as linear combination of that basis

If P were strictly positive definite, the eigenvalues would have been strictly positive

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Assume $\mathbf{x} \in \Re^n$ and $P \in \Re^{n \times n}$.

1. P is symmetric positive definite iff:
   1. Symmetric: $P^T = P$
   2. Positive Definite: $\forall \mathbf{x} \neq 0$, $\mathbf{x}^T P \mathbf{x} \geq 0$

Proof:

- If P is symmetric positive definite (SPD), then P can be written as:
  - $P = LDL^T$, where ...
    - L is lower triangular matrix with a 1 in each diagonal entry.
    - D is diagonal matrix with positive values.
- So, we can write $P = RR^T$ where $R = L\sqrt{D}$.
- Thus we have $\mathbf{x}^T P \mathbf{x} = \mathbf{x}^T RR^T \mathbf{x} = (R^T \mathbf{x})^T (R^T \mathbf{x}) = \mathbf{y}^T \mathbf{y}$
  - where $\mathbf{y} = (R^T \mathbf{x})$ and thus $\mathbf{y} \in \Re^n$.
- So, $\mathbf{x}^T P \mathbf{x} \geq 0$.

Can we use this decomp
-osition to show
other norm properties?
Triangle ineq?

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Recall:

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof:

Basic idea is to consider standard 2-norm on a transformed space xR

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Recall:

1. Normed Vector Space: A vector space on which a norm is defined. In any real vector space $\Re^n$, the length of a vector has the following properties:
   1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
   2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ for any scalar α.
   3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof:

1. By definition of PST: $\|\mathbf{x}^T P \mathbf{x}\| \geq 0$, and $\|\mathbf{x}^T P \mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
2. For any scalar α: $\|\alpha \mathbf{x}\|_P = \sqrt{(\alpha \mathbf{x})^T P (\alpha \mathbf{x})} = \sqrt{(\alpha^2)(\mathbf{x}^T P \mathbf{x})} = \alpha \sqrt{\mathbf{x}^T P \mathbf{x}} = |\alpha| \|\mathbf{x}\|_P$.
3. $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P + \|\mathbf{x}_2\|_P$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. **Next Slide.**

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Proof for $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$

Under what conditions on P is $\sqrt{\mathbf{x}^T P \mathbf{x}}$ a valid Norm?

Proof for $\|\mathbf{x}_1 + \mathbf{x}_2\|_P \leq \|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$

For any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$:

This is a more verbose proof in terms of the quadratic expansion itself, instead of 2-norm on the transformed space xR

1. $\|\mathbf{x}_1 + \mathbf{x}_2\|_P^2 =$
   - $(\mathbf{x}_1 + \mathbf{x}_2)^T P (\mathbf{x}_1 + \mathbf{x}_2)$
   - $\mathbf{x}_1^T P \mathbf{x}_1 + \mathbf{x}_2^T P \mathbf{x}_2 + \mathbf{x}_1^T P \mathbf{x}_2 + \mathbf{x}_2^T P \mathbf{x}_1$
   - $u^T u + v^T v + u^T v + v^T u$ (Using $P = RR^T$, $u = R^T \mathbf{x}_1$ and $v = R^T \mathbf{x}_2$)
   - $u^T u + v^T v + 2u^T v$, since $u^T v = v^T u$
2. $(\|\mathbf{x}_1\|_P + \|\mathbf{x}_2\|_P)^2 =$
   - $\|\mathbf{x}_1\|_P^2 + \|\mathbf{x}_2\|_P^2 + 2\|\mathbf{x}_1\|_P \|\mathbf{x}_2\|_P$
   - $\mathbf{x}_1^T P \mathbf{x}_1 + \mathbf{x}_2^T P \mathbf{x}_2 + 2\sqrt{(\mathbf{x}_1^T P \mathbf{x}_1)(\mathbf{x}_2^T P \mathbf{x}_2)}$
   - $u^T u + v^T v + 2\sqrt{(u^T u)(v^T v)}$
3. By Cauchy Schwarz Inequality: $u^T v \leq \sqrt{(u^T u)(v^T v)}$ $(Cos(\theta) \leq 1)$

Recall: Inner Product Space

It is a vector space over a field of scalars along with an inner product.

- **Field of scalars:** e.g. IR algebraic structure with:-
  1. Addition: must be multiplicative and associative.
  2. Subtraction.
  3. Multiplication: must be commutative, associative and distributive.
  4. Division: multiplicative inverse must exist.
- **Inner Product:**
  1. (Conjugate) Symmetry: $<\mathbf{x}_1, \mathbf{x}_2> = \overline{< \mathbf{x}_2, \mathbf{x}_1 >}$.
  2. Linearity in the first argument.
     - ⋆ $< a\mathbf{x}_1, \mathbf{x}_2 >= a < \mathbf{x}_1, \mathbf{x}_2 >$
     - ⋆ $< \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_3 >=< \mathbf{x}_1, \mathbf{x}_3 > + < \mathbf{x}_3, \mathbf{x}_3 >$
  3. Positive definiteness: $< \mathbf{x}, \mathbf{x} >\geq 0$, with equality iff $\mathbf{x} = 0$.

Prove that inner product space is a normed vector space.

Q) Why field of scalers?

A) By conjugate symmetry, we have $<\mathbf{x},\mathbf{x}> = \overline{<\mathbf{x},\mathbf{x}>}$. So $<\mathbf{x},\mathbf{x}>$ must be real.

So, we can define $\|\mathbf{x}\| = \sqrt{<\mathbf{x},\mathbf{x}>}$.

We need to prove that $\|\mathbf{x}\|$ is a valid norm:-

1. By positive definiteness: $<\mathbf{x},\mathbf{x}> \geq 0$, with equality iff $\mathbf{x} = 0$. So $\|\mathbf{x}\| \geq 0$ ($=$ iff $\mathbf{x} = 0$).
2. For any complex t, $\|t\mathbf{x}\| = \sqrt{<t\mathbf{x}, t\mathbf{x}>} = \sqrt{t * \bar{t} <\mathbf{x},\mathbf{x}>} = |t|\sqrt{<\mathbf{x},\mathbf{x}>}$ (as $|t| = \sqrt{t * \bar{t}}$) So $\|t\mathbf{x}\| == |t|\|\mathbf{x}\|$
3. $\|\mathbf{x}_1 + \mathbf{x}_2\| = \sqrt{<\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2>} = \sqrt{<\mathbf{x}_1,\mathbf{x}_1> + <\mathbf{x}_2,\mathbf{x}_2> + <\mathbf{x}_1,\mathbf{x}_2> + <\mathbf{x}_2,\mathbf{x}_1>}$ $\leq \sqrt{<\mathbf{x}_1,\mathbf{x}_1> + <\mathbf{x}_2,\mathbf{x}_2> + 2\sqrt{<\mathbf{x}_1,\mathbf{x}_1><\mathbf{x}_2,\mathbf{x}_2>}}$ (by Cauchy Schwartz inequality)

Only these two desired for triangle inequality

Example of normed vector space that is not an inner product space.

$$\|\mathbf{x}\|_p = \left[\ \sum_{i=1}^{\infty} |\mathbf{x}_i|^p\ \right]^{\frac{1}{p}}$$

p != 2

H/w: Argue...

Prove that $|<u,v>| \leq \|u\|_P \, \|v\|_P$ for any norm P

Proof:

- If u = 0 or v = 0, then L.H.S. = R.H.S = 0. Hence the equality holds.
- Assume u,v $\neq$ 0. Let z = u - $\frac{<u,v>}{<v,v>}$v.
- By linearity of inner product in first argument, we have:
  $<z,v> = <u - \frac{<u,v>}{<v,v>}v,v> = <u,v> - \frac{<u,v>}{<v,v>} <v,v> = 0$
- Therefore, $<u,u> = <z+\frac{<u,v>}{<v,v>}v,z+\frac{<u,v>}{<v,v>}v> = <z,z> + (\frac{<u,v>}{<v,v>})^2<v,v> + 0$
- So $<u,u> \geq \frac{|<u,v>|^2}{<v,v>}$