So far: (a) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = set of all convex combinations of pts in S

(b) $\underbrace{\text{Convex hull}(S)}_{\text{denoted conv}(S)}$ = Smallest convex set that contains S [Prove as h/w]

Also: The idea of a convex combination can be generalised to include infinite sums, integrals, and, in the most general form, probability distributions

Similarly: (a) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or } \text{aff}(S)}$ = set of all conic/affine combination of pts in S

(b) $\underbrace{\text{Conic/Affine hull}(S)}_{\text{conic}(S) \text{ or } \text{aff}(S)}$ = Smallest conic/affine set that contains S

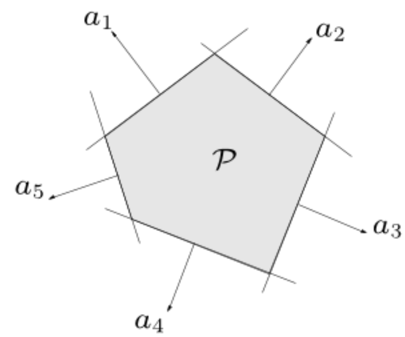
Q: Can you define convex polyhedra (polytope) in terms of convex hull? Leads to defn of simplex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Ans: If $\exists\, S \subset P$ s.t. $|S|$ is finite & $P = \text{convexhull}(S)$ then P is a polyhedron (polytope

Simplex: A n-dimensional simplex is convexhull(S) where S is affinely independent set of n+1 pts

Positive semidefinite cone

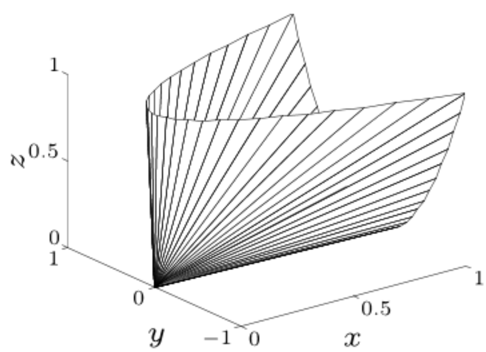
notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone
- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

the intersection of (any number of) convex sets is convex

example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$: