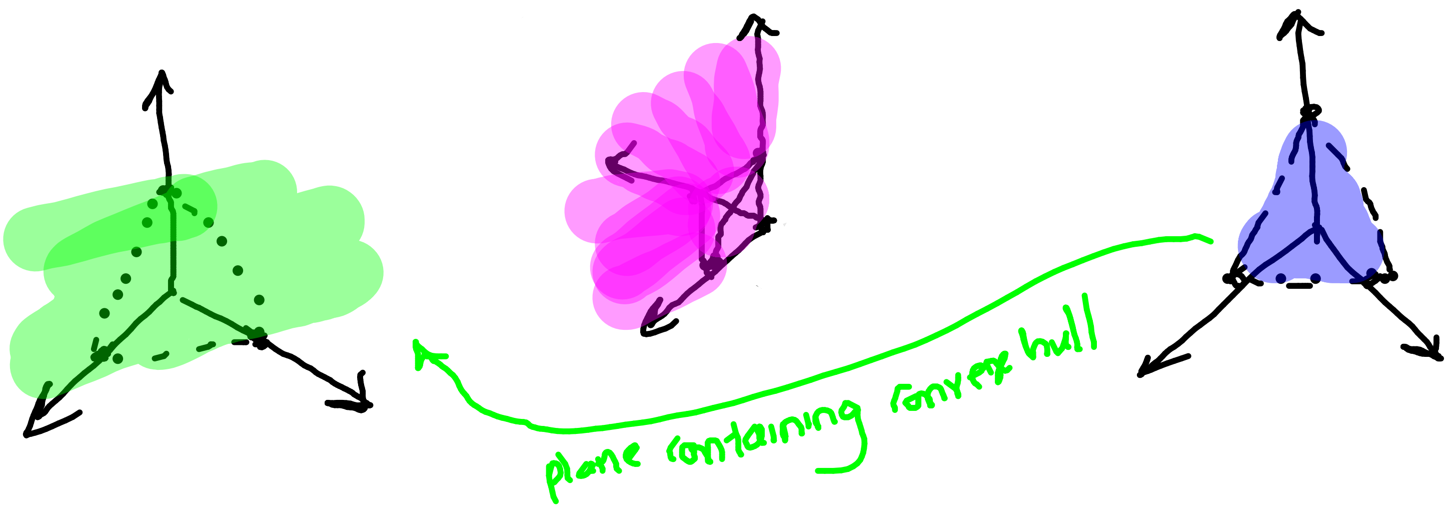
Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

what is aff hull(S)? what is conichull(S)? What is convex hull(S)



S is called conically spanning set of cone K iff conic$(S) = K$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

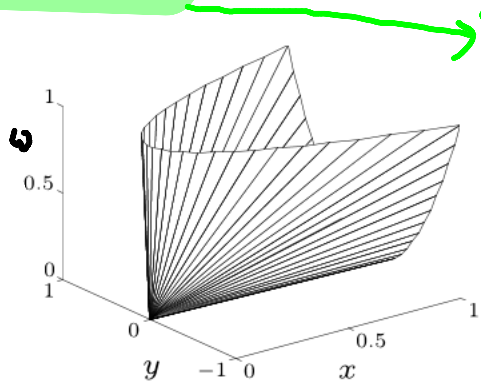
Easy to prove it is a cone

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Q: Is it convex? ✓ Is it a cone? ✗ Since $0 \notin S^n_{++}$

example: $\begin{bmatrix} x & y \\ y & \omega \end{bmatrix} \in \mathbf{S}^2_+$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

show that $\forall\, x_1, x_2 \in C$, $\forall\, 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

3. Empirical / Experimental [Homework]

Look for "smart" ideas

You may want to sample x_1 & x_2 along boundary instead of randomly

Intersection

the intersection of (any number of) convex sets is convex

↳ show that S^n_+ is convex using this property

example:

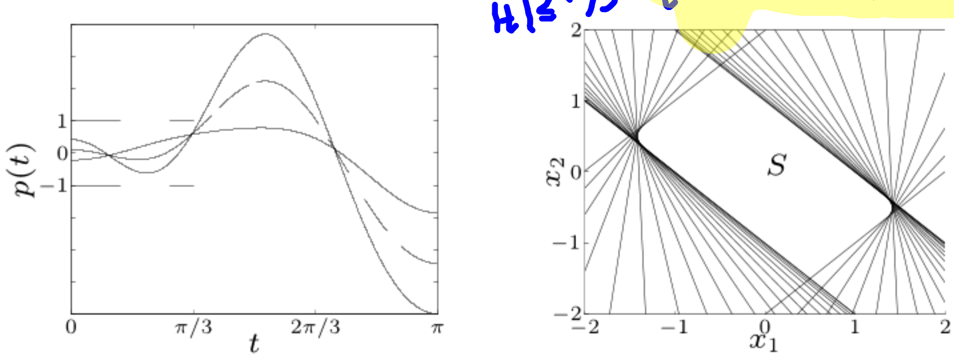
$$S = \{x \in \mathbf{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:

$S = \bigcap_{|t| \le \pi/3} \{x \in R^m \mid |\sum_{i=1}^{m} x_i \cos(it)| \le 1\}$

As shown in class, this is convex for fixed t

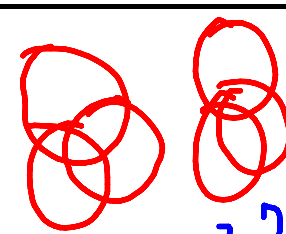


Helly's Theorem

Let C be a finite family of convex sets in $\mathbf{R}^n$ such that, for $k \le n + 1$, any k (set) members of C have a nonempty intersection. Then the intersection of all (set) members of C is nonempty.

↳ Intersection of any # of sets upto the dimension (n) of the space is non-empty

⇒ Intersection of all sets is non-empty



[Bad example]

$A = \begin{bmatrix} 1 & & & \\ & 2 & 0 & \\ & 0 & 3 & \\ & & & 4 \end{bmatrix} \rightarrow$ Rescales

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ Permutes x_1 & x_2

Affine function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$) → translation

Permutes rotates & rescales

Matrix of real scalars

- the image of a convex set under f is convex

If dom is convex range will be: $S \subseteq \mathbf{R}^n$ convex $\implies$ $f(S) = \{f(x) \mid x \in S\}$ convex

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

If range is convex domain will be: $C \subseteq \mathbf{R}^m$ convex $\implies$ $f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\}$ convex

$x^T \tilde{A} \le \tilde{b}$: An affine set (every affine set is cvx)

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$) — symmetric pxp matrices
- hyperbolic cone $\{x \mid x^T P x \le (c^T x)^2,\ c^T x \ge 0\}$ (with $P \in \mathbf{S}^n_+$)

[H/W]

inverse image

symmetric psd matrices

$x \in \mathbf{R}^n$

$A = \begin{bmatrix} P^{1/2} \\ c^T \end{bmatrix}$ $b = 0 \rightarrow$ Affine transform

$\begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} P^{1/2} x \\ c^T x \end{bmatrix}$

$\{(y,t) \mid y^T y \le t^2\}$: a second order convex cone in the image of the affine transform

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x, t) = x/t, \qquad \mathbf{dom}\, P = \{(x, t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex