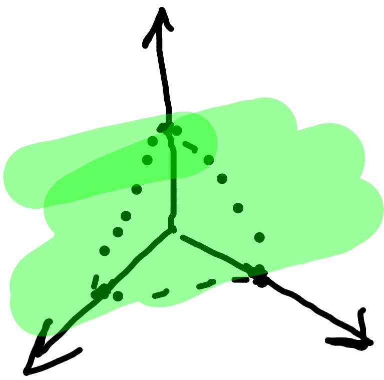
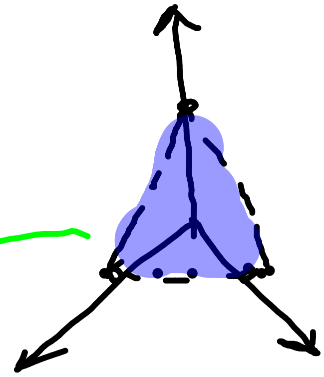
Let $S=\{(1,0,0),(0,1,0),(0,0,1)\}$

what is aff hull(S)? what is conichull(S)? What is convex hull(S)





plane containing convex hull

S is called conically spanning set of cone K iff conic(S) $= K$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices → Easy to prove it is a cone

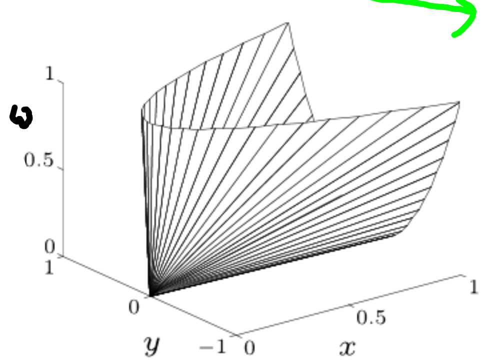
$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

$\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices → Q: Is it convex? ✓ Is it a cone? ✗

Since $0 \notin \mathbf{S}^n_{++}$

example: $\begin{bmatrix} x & y \\ y & w \end{bmatrix} \in \mathbf{S}^2_+$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

show that $\forall\, x_1, x_2 \in C, \forall\, 0 \le \theta \le 1 \implies \theta x_1 + (1-\theta)x_2 \in C$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

3. Empirical / Experimental [Homework]

Look for "smart" ideas

You may want to sample x_1 & x_2 along boundary instead of randomly

Intersection

the intersection of (any number of) convex sets is convex

↳ show that S^n_+ is convex using this property

example:

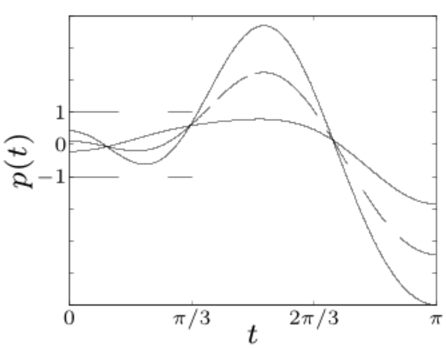
$$S = \{x \in \mathbf{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

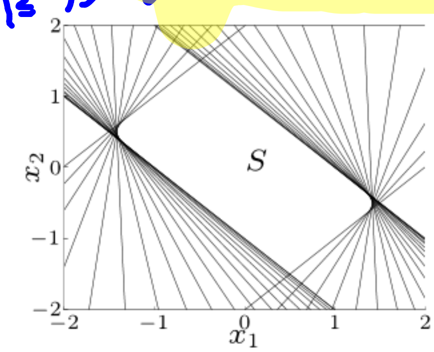
where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

$S = \bigcap_{|t| \le \pi/3} \{x \in R^m \mid |\sum_{i=1}^{m} x_i \cos(it)| \le 1\}$

As shown in class, this is convex for fixed t

for $m = 2$:



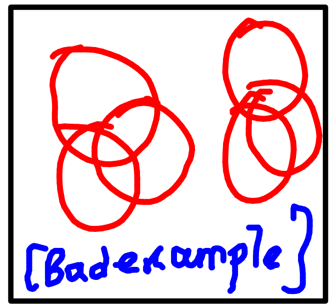


Helly's Theorem

Let C be a finite family of convex sets in $\mathbf{R}^n$ such that, for $k \leq n+1$, any k (set) members of C have a nonempty intersection. Then the intersection of all (set) members of C is nonempty.

↳ Intersection of any # of sets upto the dimension (n) of the space is non-empty

⇒ Intersection of all sets is non-empty



$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix} \to$ Rescales

$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$ Permutes x_1 & x_2

Affine function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$) → translation

permutes rotates & rescales

Matrix of real scalars

- the image of a convex set under f is convex

if dom is convex range will be: $S \subseteq \mathbf{R}^n$ convex $\implies$ $f(S) = \{f(x) \mid x \in S\}$ convex

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

if range is convex domain will be: $C \subseteq \mathbf{R}^m$ convex $\implies$ $f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\}$ convex

$x^T \tilde{A} \le \tilde{b}$: An affine set (every affine set is cvx)

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1A_1 + \cdots + x_mA_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$) — symmetric pxp matrices
- hyperbolic cone $\{x \mid x^TPx \le (c^Tx)^2,\ c^Tx \ge 0\}$ (with $P \in \mathbf{S}^n_+$)

[HW]

inverse image

symmetric psd matrices

$x \in R^n$

$\{(y,t) \mid y^Ty \le t^2\}$: a second order convex cone in the image of the affine transform

$A = \begin{bmatrix} P^{1/2} \\ c^T \end{bmatrix}$ $b = 0 \to$ Affine transform

$\begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} P^{1/2}x \\ c^Tx \end{bmatrix}$

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

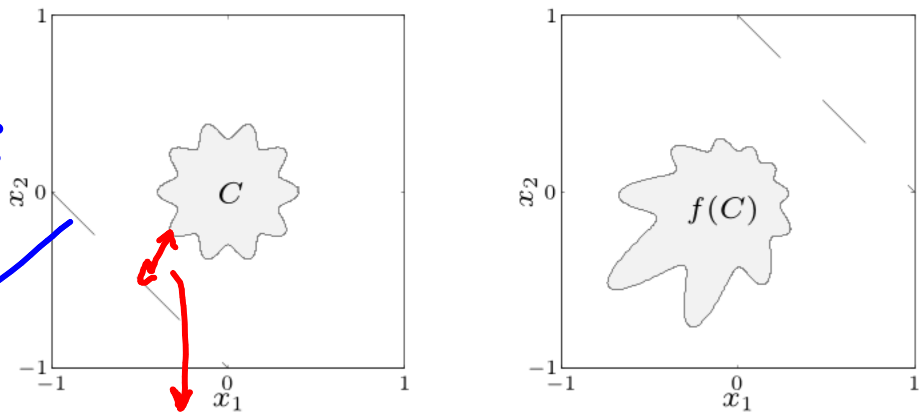
linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax + \cancel{b}}{c^Tx + d}, \qquad \mathbf{dom}\, f = \{x \mid c^Tx + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

example of a linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$



(In general, $c^T x + d = 0$:) $x_1 + x_2 + 1 = 0$

rescaling based on inverse of distance

Some topological concepts:

[Topological set is set with concept of nbrhood]

① A set U is called an open set if it does not contain any of its boundary pts. If S is a metric space (eg an inner product space) with distance metric $d(x,y)$, then a subset U of S is called open if, given any $x \in U$, $\exists\ \epsilon > 0$ such that given any $y \in S$ with $d(x,y) < \epsilon$, $y \in U$

② A set $V \subseteq S$ is called closed if its complement $S \setminus V$ is an open set

③ $x \in S$ is called an interior point of S if there exists a neighborhood of x contained in S. If S is a metric space, then $x \in S$ is an interior pt if $\exists\, \epsilon > 0$ s.t $\forall y$ s.t $d(x,y) < \epsilon$, $y \in S$

The set of all interior pts of C form the interior of C. Thus, if S is a metric space & $C \subseteq S$

$$\underbrace{\text{int}(C)}_{\text{interior of } C} = \{x \mid \exists\, \epsilon > 0 \text{ s.t } \forall y \text{ s.t } d(x,y) < \epsilon,\ y \in C\}$$

What can I say if interior$(C) = \phi$

eg: sufficient conditions

① $C \subseteq$ hyperplane. In particular, if S is affine this is necessary & sufficient

② eg: a shell

③ eg: $C = \partial K$ in topological space S (see next page for defn of boundary of set K denoted by ∂K)

subset of S

(4) The set of pts of a set C s.t every neighborhood of a point from the set consists of atleast one point in C and one point not in C is called the boundary ∂C of C. If S is a metric space

$$\partial C = \{x \in C \mid \forall \epsilon > 0, \exists y \text{ s.t } d(x,y) < \epsilon \ \& \ y \in C \text{ and } \exists y' \text{ s.t } d(x,y') < \epsilon \ \& \ y' \notin C\}$$

(5) Let S be a subset of a topological space X. A point $x \in X$ is a limit point of S if every neighborhood of x contains at least one point of S different from x itself.

can be relaxed to open neighborhoods without loss

If S happens to have an associated metric d, and $A \subseteq S$, then $\alpha \in S$ is a limit point of A iff:

$$\forall \epsilon > 0 : \{x \in A \text{ s.t } 0 < d(x,\alpha) < \epsilon\} \neq \phi$$

[Note: α need not belong to A]

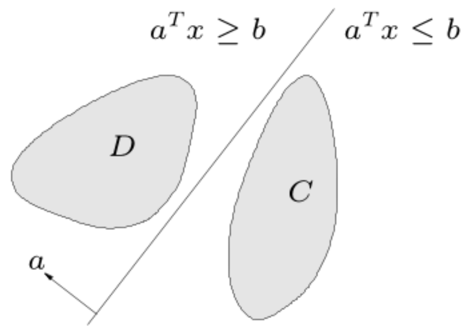
Informally speaking, is a limit point of A if there are points in A that are different from but arbitrarily close to it

(6) Closure of S [$cl(S)$] $= S \cup \{\text{limit points of } S\}$

Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

$\boxed{\text{Proof:}}$ Let $S = \{x - y \mid x \in C, y \in D\}$.

Now we can prove (see http://www.cse.iitb.ac.in/~cs709/notes/eNotes/ExtraProblems-1.pdf, Q1) that S, being a sum of two convex sets, is convex.

Since $C \cap D = \phi$, $0 \notin S$

(a) Suppose $0 \notin cl(S)$: Consider the sets $\{0\}$ and $cl(S)$. We will prove that $\exists\, a \neq 0$ s.t

$$\underbrace{a^T z > 0 \ \forall z \in cl(S)}_{\text{Q: How to choose } a?} \ \& \ \underbrace{a^T w = 0 \text{ for } w \in \{0\}}_{\text{obvious}}$$

complete proof given in class: H/w

i.e $\exists a$ s.t $a^T(x-y) > 0 \ \forall \ x-y \in S$

i.e $a^Tx > a^Ty \ \forall \ x \in C$ & $y \in D$

Let $b = \inf_{x \in C} a^Tx$. Then we proved existence of a & b s.t

$$a^Tx \geq b \ \forall \ x \in C \ \ \& \ \ a^Ty \leq b \ \forall \ y \in D$$

(b) suppose $0 \in cl(S)$. Since $0 \notin S$, $0 \in bndry(S)$

If interior $(S) = \phi$ (empty), S must be $\subseteq \{z \mid a^Tz = b\}$ A hyperplane

& the hyperplane must includ 0 on $bndry(S)$ &

$\Rightarrow b = 0$. i.e $a^Tx = a^Ty \ \forall x \in C$ & $y \in D$

$\Rightarrow$ we have a trivial separating hyperplane

If interior$(S) \neq \phi$ (non-empty), consider

$S_{-\epsilon} = \{z \mid B(z,\epsilon) \subseteq S\}$ for $\epsilon > 0$

↘ Ball with center z and radius $\epsilon > 0$.

Thus

- $S_{-\epsilon}$ is the set S shrunk by $-\epsilon$
- $cl(S_{-\epsilon})$ is closed & convex (why?) and does not contain 0 and hence, from part (a) above, it is strictly separated from $\{0\}$ by atleast one hyperplane with normal vector $a(\epsilon)$:

$$a(\epsilon)^T z > 0 \quad \forall\ z \in S_{-\epsilon}$$

- Let ϵ_k, $k = 1, 2, \ldots$ be a sequence of positive values of ϵ_k with $\lim_{k \to \infty} \epsilon_k = 0$

Let $\|a(\epsilon_k)\|_2 = 1\ \forall k$ (without loss of generality)

- Thus, the sequence $a(\epsilon_k)$ contains a convergent subsequence, and denoting its limit by $\bar{a}$, we have

$$\boxed{a(\epsilon_k)^T z > 0} \quad \forall z \in S_{-\epsilon_k}$$

for all k & therefore

$$\boxed{\bar{a}^T z > 0 \quad \forall \; z \in \text{interior}(S)}$$

proof by contradiction

and

$$\bar{a}^T z \geq 0 \quad \forall \; z \in S$$

that is

$$\bar{a}^T x \geq \bar{a}^T y$$

$$\forall x \in C \;\&\; y \in D$$

(use the property that a convex set is connected?)

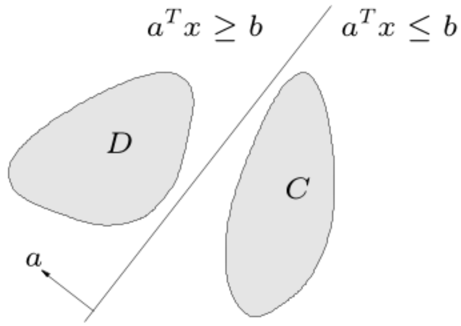
Hence proved!

Separating hyperplane theorem

Thus

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Consequence

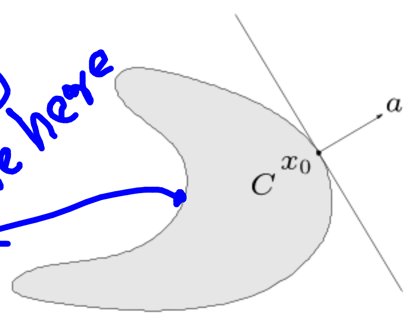
Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

no supporting hyperplane here



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Proof (from separating hyperplane theorem):

ⓐ interior$(C) \neq \phi$

Apply separating hyperplane theorem to sets $C' = \{x_0\}$ and $D' =$ interior(C)

ⓑ If interior$(C) = \phi$

$C \subseteq$ Affine set of dimension $< n$

$\Rightarrow$ Any hyperplane containing that affine set contains C & x_0

$\Rightarrow$ This hyperplane is a trivial supporting hyperplane