Let us resume our discussion on linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\le 0 \end{array}$$

Let: $\lambda \geq 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b) \le 0$

$\Rightarrow c^Tx \geq c^Tx + \lambda^T(-Ax+b)$

$= \lambda^T b + (c - A^T\lambda)^T x$

$\geq \min\limits_x \lambda^T b + (c - A^T\lambda)^T x$

$= \begin{cases} \lambda^T b & \text{if } A^T\lambda = c \\ -\infty & \text{if } A^T\lambda \neq c \end{cases}$

independent of λ

independent of x

$\Rightarrow$

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^Tx \\ \text{s.t} & Ax \geq b \end{array} \geq \begin{array}{ll} \max & b^T\lambda \\ \lambda \geq 0 & \\ \text{s.t} & A^T\lambda = c \end{array}$$

Primal LP (lower bounded)

Dual LP (upper bounded)

Conic Program (CP)

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\le_K 0 \end{array}$$

K is a regular/proper cone

Generalised cone program

$$\begin{array}{ll} \min\limits_{x\in S} & \langle c, x\rangle_S \\ \text{subject to} & Ax - b \in K \end{array}$$

We need an equivalent $\lambda \in K^*$ s.t

$\langle \lambda, Ax - b\rangle \geq 0$

This K^* s.t

$K^* = \{\lambda \mid \langle \lambda, Ax-b\rangle \geq 0 \ \forall\, Ax - b \in K\}$

is called the DUAL CONE of K

by dual) | by primal)

Called the weak duality theorem for Linear Program

$K_* = \{\lambda : \lambda^T \xi \geq 0 \,\forall \xi \in K\}$ is the cone dual to K

{defn on page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf}

With this, prove the following weak duality theorem for CONIC PROGRAM

$$\min_{x \in S} \langle c, x \rangle \text{ s.t. } Ax \geq_K b \quad \geq \quad \max_{\lambda \in K^*} \langle b, \lambda \rangle \text{ s.t. } A^T \lambda = c$$

Primal CP (lower bounded by dual)

Dual CP (upperbounded by primal)

Note: 1. Both LP & CP dealt with affine objective

2. CP dealt with the generalised conic inequalities

3. Later, in convex programs, we will deal with the more general convex functions in the objective

Notes:

1. If $K = \mathbb{R}^n_+$, the CP is an LP

If $K = S^n_+$, the CP is an SDP

S^n_+: Set of all $n \times n$ symmetric positive semidefinite matrices

SDP: Semi-definite program

2. Any generic convex program can be expressed as a cone program (CP) [H/W]

HOW ABOUT STRONG DUALITY FOR LPs?

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Theorem 1.2.2 [Duality Theorem in Linear Programming] *Consider a linear programming program*

$$\min_x \left\{ c^T x \,\middle|\, Ax \geq b \right\} \tag{LP}$$

along with its dual

$$\max_y \left\{ b^T y \,\middle|\, A^T y = c, y \geq 0 \right\} \tag{LP*}$$

Then

1) *The duality is symmetric: the problem dual to dual is equivalent to the primal;*

2) *The value of the dual objective at every dual feasible solution is $\leq$ the value of the primal objective at every primal feasible solution*

Weak LP duality (already proved)

3) *The following 5 properties are equivalent to each other:*

(i) *The primal is feasible and bounded below.*

(ii) *The dual is feasible and bounded above.*

(iii) *The primal is solvable.*

(iv) *The dual is solvable.*

(v) *Both primal and dual are feasible.*

Whenever (i) $\equiv$ (ii) $\equiv$ (iii) $\equiv$ (iv) $\equiv$ (v) is the case, the optimal values of the primal and the dual problems are equal to each other. : Strong duality = [illegible] + (3)

H/W: Prove (1) & (3)

Theorem 1.2.3 [Necessary and sufficient optimality conditions in linear programming] *Consider an LP program* (LP) *along with its dual* (LP*). *A pair* (x, y) *of primal and dual feasible solutions is comprised of optimal solutions to the respective problems if and only if*

$$y_i[Ax - b]_i = 0, \; i = 1, ..., m, \qquad \text{[complementary slackness]}$$

likewise as if and only if

$$c^T x - b^T y = 0 \qquad \text{[zero duality gap]}$$

special case of Karush Kuhn Tucker (KKT) conditions to be discussed later

Proof sketch: [H/W complete the proof rigorously]

$\boxed{\text{Only if}}$ From Theorem 1.2.2, if x & y are pts of optimal primal & dual solns respectively, then

$$A^T y = c \Rightarrow (A^T y)^T x = c^T x = y^T b \Rightarrow y^T(Ax - b) = 0$$

$$\Rightarrow \forall i, \; y_i [Ax - b]_i = 0$$

$\boxed{\text{if}}$ $y_i[Ax - b]_i = 0 \Rightarrow y^T(Ax - b) = 0 \Rightarrow y^T b = y^T A x$

$\Rightarrow$ Dual is solvable (condition 3.(iv) of Theorem 1.2.2)

$\Rightarrow$ Conditions (1) & (3) of Theorem 1.2.2 are met

can be proved independently

Similar Duality theorem for CP:

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$$\min_x \left\{ c^T x : \underbrace{Ax - b \geq_K 0}_{\Leftrightarrow Ax - b \in K} \right\} \qquad \text{(CP)}$$

$$\max_\lambda \left\{ b^T \lambda : A^T \lambda = c, \lambda \geq_{K_*} 0 \right\}, \qquad \text{(D)}$$

Theorem 2.1. *Assuming A in* (CP) *is of full column rank, the following is true:*

(i) *The duality is symmetric:* (D) *is a conic problem, and the conic dual to* (D) *is (equivalent to)* (CP);

(ii) [weak duality] Opt(D) $\leq$ Opt(CP); → Already proved

(iii) [strong duality] *If one of the programs* (CP), (D) *is bounded and strictly feasible* (i.e., the corresponding affine plane intersects the interior of the associated cone), *then the other is solvable and* Opt(CP) = Opt(D). *If both* (CP), (D) *are strictly feasible, then both programs are solvable and* Opt(CP) = Opt(D);

(iv) [optimality conditions] *Assume that both* (CP), (D) *are strictly feasible. Then a pair* (x, λ) *of feasible solutions to the problem is comprised of optimal solutions iff* $c^T x = b^T \lambda$ *("zero duality gap"), same as iff* $\lambda^T[Ax - b] = 0$ *("complementary slackness")*.

H/W: Prove these in a manner similar to the duality theorem for LP

Note: Duality theorems for LP and CP are special cases of Lagrange duality that we will discuss later in the course

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

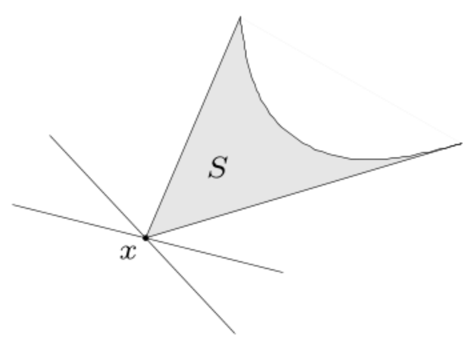
dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

Minimum and minimal elements via dual inequalities

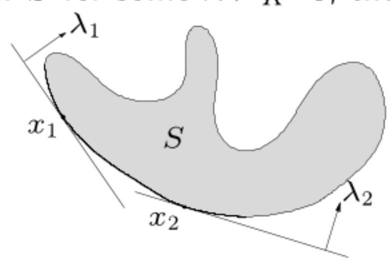
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

Thus (continuing our story of dual descriptions of sets)

3) if C = conic set $\subseteq S$ and B is its basis

$$C = \text{conichull}(B) = \{ s \in S \mid \langle s, b \rangle_S \geq 0 \ \forall\, b \in B^* \}$$

where B^* is basis for $C^* = \{ s^* \in S \mid \langle s^*, c \rangle_S \geq 0 \ \forall\, c \in C \}$