Discussion on homework problem:

16/08/2013 Prove that if S is convex and its closure does not contain 0, then there exists a hyperplane that strictly separates S from {0}.
Deadline: August 21 2013.

Soln on page 46 of Boyd's book: http://www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

eral case as an exercise (exercise 2.22). We assume that the (Euclidean) *distance* between C and D, defined as

$$\mathbf{dist}(C, D) = \inf\{\|u - v\|_2 \mid u \in C,\ v \in D\},$$

is positive, and that there exist points $c \in C$ and $d \in D$ that achieve the minimum distance, *i.e.*, $\|c - d\|_2 = \mathbf{dist}(C, D)$. (These conditions are satisfied, for example, when C and D are closed and one set is bounded.)

Define

$$a = d - c, \qquad b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}.$$

We will show that the affine function

$$f(x) = a^T x - b = (d - c)^T (x - (1/2)(d + c))$$

is nonpositive on C and nonnegative on D, *i.e.*, that the hyperplane $\{x \mid a^T x = b\}$ separates C and D. This hyperplane is perpendicular to the line segment between c and d, and passes through its midpoint, as shown in figure 2.20.

We first show that f is nonnegative on D. The proof that f is nonpositive on C is similar (or follows by swapping C and D and considering $-f$). Suppose there were a point $u \in D$ for which

$$f(u) = (d - c)^T (u - (1/2)(d + c)) < 0. \qquad (2.16)$$

We can express $f(u)$ as

$$f(u) = (d - c)^T (u - d + (1/2)(d - c)) = (d - c)^T (u - d) + (1/2)\|d - c\|_2^2.$$

We see that (2.16) implies $(d - c)^T (u - d) < 0$. Now we observe that

$$\left.\frac{d}{dt}\|d + t(u - d) - c\|_2^2\right|_{t=0} = 2(d - c)^T (u - d) < 0, \qquad a$$

so for some small $t > 0$, with $t \leq 1$, we have

$$\|d + t(u - d) - c\|_2 < \|d - c\|_2, \qquad \text{(b)}$$

(Note (a) $\Rightarrow$ (b) because $f(t) = \|d + t(u-d) - c\|^2$ is continuous & differentiable around $t=0$ & because of the properties of $f'(0) < 0$ & decreasing nature of such a function as explained on page 8 of

http://www.cse.iitb.ac.in/~CS709/notes/BasicsOfConvexOptimization.pdf reproduced below)

Theorem 46 *Let $\mathcal{I}$ be an interval and suppose f is continuous on $\mathcal{I}$ and differentiable on $int(\mathcal{I})$. Then:*

1. *if $f'(x) > 0$ for all $x \in int(\mathcal{I})$, then f is increasing on $\mathcal{I}$;*
2. *if $f'(x) < 0$ for all $x \in int(\mathcal{I})$, then f is decreasing on $\mathcal{I}$;*
3. *if $f'(x) = 0$ for all $x \in int(\mathcal{I})$, iff, f is constant on $\mathcal{I}$.*

Proof: Let $t \in \mathcal{I}$ and $x \in \mathcal{I}$ with $t < x$. By virtue of the mean value theorem, $\exists c \in (t, x)$ such that $f'(c) = \frac{f(x)-f(t)}{x-t}$.

- If $f'(x) > 0$ for all $x \in int(\mathcal{I})$, $f'(c) > 0$, which implies that $f(x) - f(t) > 0$ and we can conclude that f is increasing on $\mathcal{I}$.
- If $f'(x) < 0$ for all $x \in int(\mathcal{I})$, $f'(c) < 0$, which implies that $f(x) - f(t) < 0$ and we can conclude that f is decreasing on $\mathcal{I}$.

- If $f'(x) = 0$ for all $x \in int(\mathcal{I})$, $f'(c) = 0$, which implies that $f(x) - f(t) = 0$, and since x and t are arbitrary, we can conclude that f is constant on $\mathcal{I}$.

i.e., the point $d + t(u - d)$ is closer to c than d is. Since D is convex and contains d and u, we have $d + t(u - d) \in D$. But this is impossible, since d is assumed to be the point in D that is closest to C.

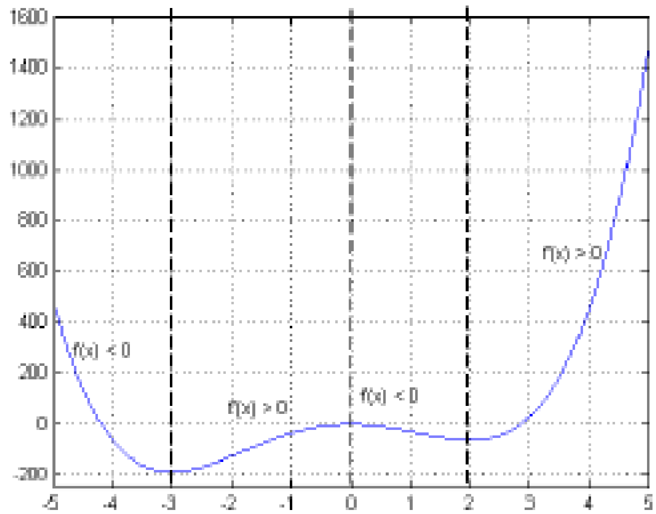


Figure 4.5: Illustration of the increasing and decreasing regions of a function $f(x) = 3x^4 + 4x^3 - 36x^2$

Linear (in)equalities, Linear programming
Conic programming

1) Recall that if S = linear vector space $\subseteq V$ & B is its basis, $S = \text{linear span}(B) = \{v \in V \mid \langle v, b\rangle_V = 0 \ \forall b \in B^\perp\}$ where $B^\perp$ is basis for $S^\perp$

Assuming V is an inner product space

eg: if $V = \mathbb{R}^n$ & $\langle a, b\rangle = a^T b$

$$\{v \in \mathbb{R}^n \mid \langle v, b\rangle_V = 0 \ \forall b \in B^\perp\}$$

$$\text{(can be written as)} \equiv \{v \in \mathbb{R}^n \mid Pv = 0\}$$

s.t $\text{rank}(P) + \dim(S) = n$

2) Recall that if A = affine set $\subseteq V$ & B is its basis, $A = \text{affine span}(B) = \{v \in V \mid \langle v, b\rangle_V = c_b \ \forall b \in B^\perp\}$ where $B^\perp$ is basis for $S^\perp$ where $A = a + S$

eg: If $V = \mathbb{R}^n$ & $\langle a, b\rangle = a^T b$

$$\{v \in \mathbb{R}^n \mid \langle v, b\rangle_V = c \ \forall b \in B^\perp\}$$

$$\text{(can be written as)} \equiv \{v \in \mathbb{R}^n \mid Pv = c\}$$

s.t $\text{rank}(P) + \dim(A) = n$

Q: What about dual representations of conic sets?

We will motivate through linear programming (LP) dual of LP & generalised inequalities:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to } -Ax+b \le 0$$

→ LINEAR PROGRAM (Recall applet: http://www.olivierdubois.info/projects/CompGeo/project/APPLET/applet.html)

Can be rewritten as $Ax \geq b$ or $Ax-b \in \mathbb{R}^n_+$

Note: $\mathbb{R}^n_+$ is a CONE. How abt defining generalised inequality for a cone K as: $c \geq_K d$ iff $c-d \in K$ and a general conic program as:

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{subject to } -Ax+b \le_K 0$$

→ CONIC PROGRAM

That is, $Ax-b \in K$

Generalized inequalities

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

→ Also referred to as a regular cone

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Some restrictions on K that we will require. H/w: WHY?

i.e. K has no st. lines passing thru O

i.e. if $a, -a \in K$, then $a = 0$

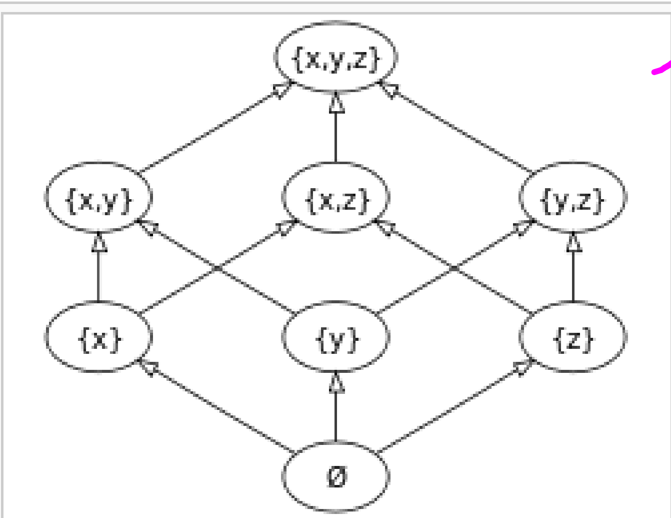
examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0,1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0,1]\}$$

To prove that K being closed, solid & pointed are necessary & sufficient conditions for $\geq_K$ to be a valid inequality, recall that any partial order $\geq$ should satisfy the following properties (refer page 51 of www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf)

1. *Reflexivity:* $a \geq a$;
2. *Anti-symmetry:* if both $a \geq b$ and $b \geq a$, then $a = b$;
3. *Transitivity:* if both $a \geq b$ and $b \geq c$, then $a \geq c$;
4. *Compatibility with linear operations:*
   (a) *Homogeneity:* if $a \geq b$ and λ is a nonnegative real, then $\lambda a \geq \lambda b$ ("One can multiply both sides of an inequality by a nonnegative real")
   (b) *Additivity:* if both $a \geq b$ and $c \geq d$, then $a + c \geq b + d$ ("One can add two inequalities of the same sign").



The Hasse diagram of the set of all subsets of a three-element set {x, y, z}, ordered by inclusion.

example partial order $\subseteq$ over sets (source: http://en.wikipedia.org/wiki/Partially_ordered_set)

that is, the $\subseteq$ partial order

generalized inequality defined by a proper cone K:

$$x \preceq_K y \iff y - x \in K, \qquad x \prec_K y \iff y - x \in \mathbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \iff x_i \le y_i, \quad i = 1, \dots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\le$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

$$y \in S \implies x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \implies y = x$$

example ($K = \mathbf{R}^2_+$)

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2