HW: In $\Re^n$, why $\|\mathbf{u}\|_p$ may not have an inner product for $p \neq 2$?

Motivation:

- Consider the following inner product on $\Re^2$: For any $\mathbf{x}, \mathbf{y} \in \Re^2$, let $< \mathbf{x}, \mathbf{y} >= 2x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$. It can be easily verified that this in an inner product (by checking for linearity, symmetry and positive definiteness by expressing it as a sum of squares).
- This inner product is certainly different from the conventional (Eucledian) dot product $< \mathbf{x}, \mathbf{y} >_E = x_1y_1 + x_2y_2$ which corredponds to the $\|.\|_2$ norm.
- Is it possible that the $< \mathbf{x}, \mathbf{y} >$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

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- Is it possible that the $<\mathbf{x}, \mathbf{y}>$ defined in step 1 (or some other such inner product) corresponds to $\|.\|_p$ norm for $p \neq 2$?

In $\Re^n$, it can be proved that for any inner product vector space $(\mathcal{V}, <.,.>)$, the inner product $<.,.>$ (including the Eucledian one) can be represented as

$$<\mathbf{u}, \mathbf{v}> = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j <\mathbf{e}_i, \mathbf{e}_j> = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}^T E \mathbf{b} = <\mathbf{a}^T, \mathbf{b}>_E$$

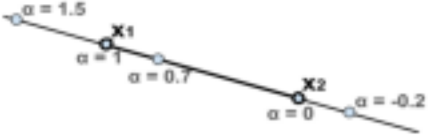
Convex Sets

Convex sets

- affine and convex sets.
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

- **In 2D, a line** through any two distinct points $\mathbf{x}_1, \mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.



  $\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$ where $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$
- In general, A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in V\ \}$.
- For some P with rank $= n - dim(V)$ and $\mathbf{b}$, A is affine iff:
  $A = \{\mathbf{x} | P\mathbf{x} = b\}$ i.e. solution set of linear equations represented by $P\mathbf{x} = b$.
  - No Solution: $\mathbf{x} = \phi$. Is that affine?
  - Unique Solution: $\mathbf{x}$ is a point.
  - Infinitely Many Solutions: $\mathbf{x}$ is a line, or a plane, etc.

Homework: Making use of the basic idea of solving linear system of equations, show the following

(conversely every affine set can be expressed as solution set of system of linear equations)

Convex set

- **In 2D, a line segment** between distinct points $\mathbf{x}_1$,$\mathbf{x}_2$: That is, all points $\mathbf{x}$ s.t.

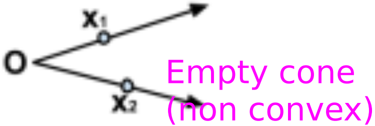
$$\begin{aligned} \mathrm{x} = \quad & \alpha x_1 + \beta x_2 \\ \text{where} \quad & \alpha + \beta = 1, 0 \leq \alpha \leq 1 (\textit{also}, 0 \leq \beta \leq 1). \end{aligned}$$

- **Convex set** : $\mathbf{x}_1, \mathbf{x}_2 \in C, 0 \leq \alpha \leq 1 \Rightarrow \alpha \mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2 \in C$

  - Convex set is connected. Convex set can but not necessarily contains 'O'

Is every affine set convex? Is the reverse true?

Cone, conic combination and convex cone



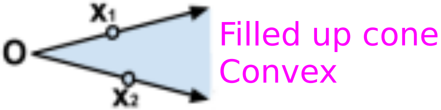
- **Cone** A set C is a cone if $\forall \mathbf{x} \in C$, $\alpha\mathbf{x} \in C$ for $\alpha \geq 0$.
- **Conic (nonnegative) combination** of points $\mathbf{x}_1, \mathbf{x}_2$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$$

$$\text{with} \quad \alpha, \beta \geq 0.$$

Example : Diagonal vector of a parallelogram is a conic combination of the two vectors (points) $\mathbf{x}_1$ and $\mathbf{x}_2$ forming the sides of the parallelogram.

- **Convex cone:** The set that contains all conic combinations of points in the set.

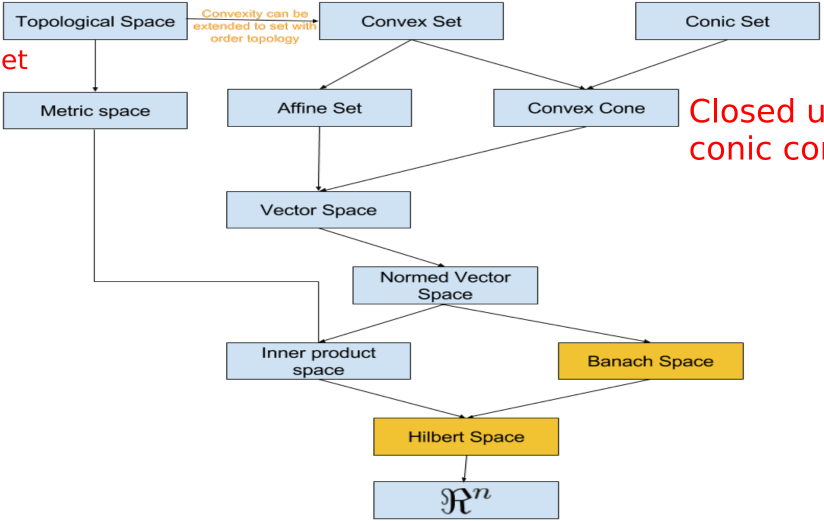


Homework: Structure of Mathematical Spaces Discussed (arrow means 'instance')

Closed under convex combinations

Has 0 mandated

Open set



Closed under conic combinations

Convex combination and convex hull

- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- **Convex hull or conv(S)** is the set of all convex combinations of point in the set S.

- Should S be always convex? No
- What about the convexity of conv(S)? Yes

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  -  

- Should S be always convex? No.
- What about the convexity of conv(S)? It's always convex.

More Convex Sets (illustrated in $\Re^n$)

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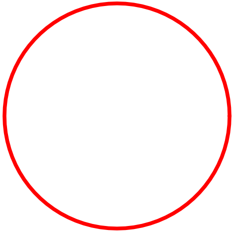
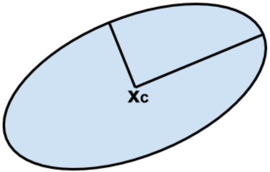
- Euclidean balls and ellipsoids.
- Norm balls and norm cones.
- Compact representation of vector space.
- Dual Representation.
- Different Representations of Affine Sets



Euclidean balls and ellipsoids

- **Euclidean ball** with **center** $\mathbf{x}_c$ and **radius** r is given by:
  $B(\mathbf{x}_c, r) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\} = \{\mathbf{x}_c + ru \mid \|u\|_2 \leq 1 \}$ Show that this set is convex
- **Ellipsoid** is a **set** of form:
  $\{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1 \}$, where P $\in S^n_{++}$ i.e. P is SPD matrix.
  - Other representation: $\{\mathbf{x}_c + \text{A u} \mid \|u\|_2 \leq 1\}$ with A square and non-singular(i.e. A^{-1} exists).



Norm balls

- **Recap Norm:** A function[6] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?

ANS: triangle inequality that is used to prove that Eucledian ball is convex set... can be simply reused

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  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) =$

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  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$,

Homework: Try and make some sense of vector induced matrix norms