Let us resume our discussion on linear programs (LP), dual of LP, conic programs & their duals

[Ref page 5 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf]

LP

Affine objective

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\leq 0 \end{array}$$

Conic Program (CP)

$$\begin{array}{ll} \min\limits_{\mathbf{x}\in\Re^n} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & -A\mathbf{x}+\mathbf{b}\leq_K 0 \end{array}$$

K is a regular | proper cone

Generalised cone program

$$\begin{array}{ll} \min\limits_{x\in S} & \langle c, x\rangle_S \\ \text{subject to} & Ax-b\in K \end{array}$$

Let: $\lambda \geq 0$ (i.e $\lambda \in \mathbb{R}^n_+$)

Then $\lambda^T(-Ax+b)\leq 0$

$$\Rightarrow c^Tx \geq c^Tx+\lambda^T(-Ax+b)$$
$$= \lambda^Tb+(c-A^T\lambda)^Tx$$
$$\geq \min_x \lambda^Tb+(c-A^T\lambda)^Tx$$
$$= \begin{cases} \lambda^Tb & \text{if } A^T\lambda=c \\ -\infty & \text{if } A^T\lambda\neq c \end{cases}$$

independent of λ

independent of x

$\Rightarrow$

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^Tx \\ \text{s.t} & Ax\geq b \end{array} \quad \geq \quad \begin{array}{ll} \max & b^T\lambda \\ \lambda\geq 0 & \\ \text{s.t} & A^T\lambda=c \end{array}$$

Primal LP (lower bounded)

Dual LP (upper bounded)

We need an equivalent $\lambda\in K^*$ s.t

$$\langle\lambda, Ax-b\rangle\geq 0$$

This K^* s.t

$$K^*=\{\lambda \mid \langle\lambda, Ax-b\rangle\geq 0 \ \forall Ax-b\in K\}$$

is called the DUAL CONE of K

by dual)

by primal)

Called the weak duality theorem for Linear Program

$K_* = \{\lambda : \lambda^T \xi \geq 0 \,\forall \xi \in K\}$ is the cone dual to K

{defn on page 7 of http://www2.isye.gatech.edu/~nemirovs/ICMNemirovski.pdf}

With this, prove the following weak duality theorem for CONIC PROGRAM

$$\min_{x \in S} \langle c, x \rangle \text{ s.t. } Ax \geq_K b \quad \geq \quad \max_{\lambda \in K^*} \langle b, \lambda \rangle \text{ s.t. } A^T\lambda = c$$

Primal CP (lower bounded by dual)

Dual CP (upperbounded by primal)

Note: ① Both LP & CP dealt with affine objective

② CP dealt with the generalised conic inequalities

③ Later, in convex programs, we will deal with the more general convex functions in the objective

Notes:

① If $K = \mathbb{R}^n_+$, the CP is an LP

If $K = S^n_+$, the CP is an SDP

Set of all $n \times n$ symmetric positive semidefinite matrices

Semi-definite program

② Any generic convex program can be expressed as a cone program (CP) [H/W]

HOW ABOUT STRONG DUALITY FOR LPs?

[Page 21 of http://www2.isye.gatech.edu/~nemirovs/Lect_ModConvOpt.pdf]

Theorem 1.2.2 [Duality Theorem in Linear Programming] *Consider a linear programming program*

$$\min_x \left\{ c^T x \,\middle|\, Ax \geq b \right\} \tag{LP}$$

along with its dual

$$\max_y \left\{ b^T y \,\middle|\, A^T y = c, y \geq 0 \right\} \tag{LP*}$$

Then

1) *The duality is symmetric: the problem dual to dual is equivalent to the primal;*

2) *The value of the dual objective at every dual feasible solution is $\leq$ the value of the primal objective at every primal feasible solution*

weak LP duality (already proved)

3) *The following 5 properties are equivalent to each other:*

$\exists\, x$ s.t $Ax \geq b$

(i) *The primal is feasible and bounded below.* $-\infty < LP$

(ii) *The dual is feasible and bounded above.* $LP^* < \infty$

(iii) *The primal is solvable.* $\exists\, \lambda$ s.t $A^T\lambda = c$

(iv) *The dual is solvable.* LP has a soln

(v) *Both primal and dual are feasible.* LP* has a soln

Whenever (i) $\equiv$ (ii) $\equiv$ (iii) $\equiv$ (iv) $\equiv$ (v) is the case, the optimal values of the primal and the dual problems are equal to each other. : Strong duality = [illegible] + (3)

H/W: Prove (1) & (3)