Norm balls

- **Recap Norm:** A function[6] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$,

Norm balls

- **Recap Norm:** A function[6] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P\mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s\in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$,

Norm balls

- **Recap Norm:** A function[6] $\|.\|$ that satisfies:
  1. $\|\mathbf{x}\| \geq 0$, and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
  2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for any scalar $\alpha \in \Re$.
  3. $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for any vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set. Why?
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup\limits_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

  Here, $\sup\limits_{s \in S} f(s) = \widehat{f}$ if $\widehat{f}$ is the minimum upper bound for $f(s)$ over $s \in S$.
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max\limits_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$, $M_N(A) = \max\limits_i \sum_{j=1}^{m} |a_{ij}|$ Homework

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation:

$$N = \|.\|_1, \; M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} \||x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then

$$N = \|.\|_1, \; M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_i|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij} x_j| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} \||x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_{j} \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x}$

$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} \|x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then

The upper bound in (3) is indeed attained at this choice of x

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} \|x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then $\|\mathbf{x}\|_1 = 1$ and $\|A\mathbf{x}\|_1 = C$
5. Thus, there exists $\mathbf{x} = [0, 0..1, 0...0]$ for which the inequalities in steps (2) and (3) become equalities! That is,

$$N = \|.\|_1,\ M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$$

1. If $N(\mathbf{x}) = \sum_{i=1}^{m} |x_j|$ then $N(A\mathbf{x}) = \sum_{i=1}^{n} |\sum_{j=1}^{m} a_{ij}x_j| \leq \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}||x_j|$
2. Changing the order of summation: $N(A\mathbf{x}) \leq \sum_{j=1}^{m}\sum_{i=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{m} \||x_j| \sum_{i=1}^{n} |a_{ij}|$
3. Let $C = \max_j \sum_{i=1}^{n} |a_{ij}| = \sum_{i=1}^{n} |a_{ik}|$. Then $\|A\mathbf{x}\|_1 \leq C\|\mathbf{x}\|_1 \Rightarrow \|A\|_1 = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \leq C$
4. Now consider a $\mathbf{x} = [0, 0..1, 0...0]$ which has 1 only in the k^{th} position and a 0 everywhere else. Then $\|\mathbf{x}\|_1 = 1$ and $\|A\mathbf{x}\|_1 = C$
5. Thus, there exists $\mathbf{x} = [0, 0..1, 0...0]$ for which the inequalities in steps (2) and (3) become equalities! That is,

$$M_N(A) = \|A\mathbf{x}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[7]):

we know that A^TA is positive semi-definite

[7] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[7]): $A^T A \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition,

[7] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[7]): $A^TA \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of A^TA such that $A^TA = U^T\Sigma U$ with $(A^TA)\mathbf{u}_i = \sigma_i\mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} =$

[7] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[7]): $A^T A \in S^n_+$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of $A^T A$ such that $A^T A = U^T \Sigma U$ with $(A^T A)\mathbf{u}_i = \sigma_i \mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$
6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^T A\mathbf{x})} =$

[7] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

If $N = \|.\|_2$, $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$

1. $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$. We know that $\|A\mathbf{x}\|_2 = \sqrt{(A\mathbf{x})^T(A\mathbf{x})} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$.
2. (From basic notes on Linear Algebra[7]): $A^T A \in S_+^n$ is symmetric positive semi-definite
3. By spectral decomposition, there exists orthonormal U with column vectors $\mathbf{u}_i$ and diagonal matrix Σ of non-negative eigenvalues σ_i of $A^T A$ such that $A^T A = U^T \Sigma U$ with $(A^T A)\mathbf{u}_i = \sigma_i \mathbf{u}_i$
4. Without loss of generality, let $\sigma_1 \geq \sigma_2 .. \geq \sigma_n$.
5. Since columns of U form an orthonormal basis for $\Re^n$, let $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$
6. Then, $\|\mathbf{x}\|_2 = \sqrt{\sum_i \alpha_i^2}$ and $\|A\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T(A^T A\mathbf{x})} = \sqrt{(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i)^T (\sum_{i=1}^{n} \sigma_i \alpha_i \mathbf{u}_i)}$.
7. If $\alpha_1 = 1$ and $\alpha_j = 0$ for all $j \neq 1$, the maximum value in (7) will be attained. Thus, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$

[7] https://www.cse.iitb.ac.in/~cs709/notes/LinearAlgebra.pdf

Norm balls: Summary

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius** r: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$ is a convex set.
  - Eg 1: **Ellipsoid** is defined using $\|\mathbf{x}\|_P^2 = \mathbf{x}^T P \mathbf{x}$.
  - Eg 2: **Euclidean ball** is defined using $\|\mathbf{x}\|_2$.
- Matrix Norm induced by vector norm N: $M_N(A) = \sup_{\mathbf{x} \neq 0} \frac{N(A\mathbf{x})}{N(\mathbf{x})}$
  - Eg: $M_N(I) = M_N(A) = 1$ irrespective of N
  - If $N = \|.\|_1$, $M_N(A) = \max_j \sum_{i=1}^{n} |a_{ij}|$
  - If $N = \|.\|_2$, $M_N(A) = \sqrt{\sigma_1}$, where σ_1 is the dominant eigenvalue of $A^T A$
  - If $N = \|.\|_\infty$, $M_N(A) = \max_i \sum_{j=1}^{m} |a_{ij}|$
- Matrix norm with an inner product: $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{trace(A^T A)}$ is the Frobenius norm.

HW: Dual Representation

If vector space $V \subseteq \Re^n$ and $\{\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_K\}$ is finite spanning set in $V^\perp$, then:-

- $V = (V^\perp)^\perp = \{\mathbf{x} | \mathbf{q}_i^T \mathbf{x} = 0; i = 1, ..., K\}$, where $K = dim(V)$
- A dual representation of vector subspace V (in $\Re^n$): $\{\mathbf{x} | Q\mathbf{x} = 0;\ \mathbf{q}_i^T$ is the i^{th} row of $Q\}$
- What about dual representations for Affine Sets, Convex Sets, Convex Cones, etc?

HW: Dual Representations of Affine Sets

Recall affine sets(say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in V\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $V(= A \text{ shifted by } -\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector space which has a dual representation $\{\mathbf{x} | Q\mathbf{x} = 0\}$
- The dual representation for A is therefore $\{x \mid Qx = Qu\}$

HW: Dual Representations of Affine Sets

Recall affine sets(say $A \subseteq \Re^n$).

- A is affine iff $\forall \mathbf{u}, \mathbf{v} \in A$: $\theta \mathbf{u} + (1-\theta)\mathbf{v} \in A, \forall\ \theta \in \Re$.
- For some vector space $V \subseteq \Re^n$, A is affine iff:
  $A(= V \text{ shifted by } \mathbf{u}) = \{\ \mathbf{u} + \mathbf{v} | \mathbf{u} \in \Re^n \text{ is fixed and } \mathbf{v} \in V\ \}$.
- Procedure: Let $\mathbf{u}$ be some element in the affine set A. Then $V(= A \text{ shifted by } -\mathbf{u}) = \{\ \mathbf{v} - \mathbf{u} | \mathbf{v} \in A\ \}$ is a vector space which has a dual representation $\{\mathbf{x} | Q\mathbf{x} = 0\}$
- The dual representation for A is therefore $\{\mathbf{x} | Q\mathbf{x} = Q\mathbf{u}\}$

HW: Dual Representations of Affine Sets

- For some Q with $rank = n - dim(V)$ and $\mathbf{u}$, A is affine iff: $A = \{\mathbf{x} | Q\mathbf{x} = Q\mathbf{u}\}$ i.e. solution set of linear equations represented by $Q\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = Q\mathbf{u}$.
- Example: In 3-d if Q has rank 1, we will get either a plane as solution or no solution. If Q has rank 2, we can get a plane, a line or no solution.
- Thus hyperplanes are affine spaces of dimension $n-1$ with $Q\mathbf{x} = \mathbf{b}$ given by $p^T\mathbf{x} = \mathbf{b}$. We will soon see the duality of convex cones, and in general convex sets.

Examples of Convex Cones

More on Convex Sets and Cones

- Half-spaces as cones (induced by hyperplanes)
- Norm Cones
- Positive Semi-definite cone.
- Positive Semi-definite cone: Example and Notes.
- Convexity Preserving Operations on Sets

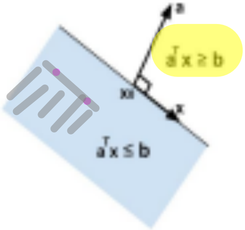
Hyperplanes and halfspaces.

Hyperplane: Set of the form $\{\mathbf{x} | \mathbf{a}^T\mathbf{x} = \mathbf{b}\}$ $(\mathbf{a} \neq 0)$

- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$
- Alternatively: $\{\mathbf{x} | (\mathbf{x} - \mathbf{x}_0) \perp \mathbf{a}\}$, where $\mathbf{a}$ is normal and $\mathbf{x}_0 \in H$

Hyperplanes and halfspaces.

halfspace: Set of the form $\{\mathbf{x}|\mathbf{a}^T\mathbf{x} \leq \mathbf{b}\}$ $(\mathbf{a} \neq 0)$

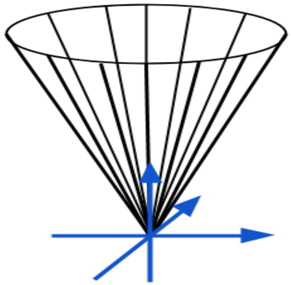


Is the half space a convex cone?
Yes: The upper half space,
as long as the hyperplane
passes through the origin..
b = 0

- where $\mathbf{b} = \mathbf{x}_0^T\mathbf{a}$

Norm cones

- **Norm ball** with **center** $\mathbf{x}_c$ and **radius r**: $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_x\| \leq r\}$.
- **Norm cone:** A **set** of form: $\{(\mathbf{x}, t) \in \Re^{n+1} | \|\mathbf{x}\| \leq t\}$.
  - Norm balls and cones are convex.
  - Euclidean norm cone is called-second order cone. If $\mathbf{x} \in \Re^2$, it is shown in $\Re^3$ as:-



Positive semidefinite cone

Notation

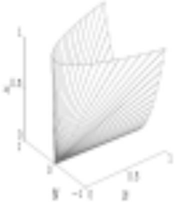
- S^n is set of symmetric $n \times n$ matrices.
- $S_+^N = \{\mathrm{X} \in S^n \mid \mathrm{X} \succeq 0\}$: positive semidefinite $n \times n$ matrices.
  - $\mathrm{X} \in S_+^N \iff z^T X z \geq 0$ for all z
  - S_+^N is a convex cone.
- $S_{++}^N = \{\mathrm{X} \in S^n \mid \mathrm{X} \succ 0\}$: positive definite $n \times n$ matrices.

Positive semidefinite cone: Example

Consider a positive semi-definite matrix S in $\Re^2$. Then S must be of the form

$$S = \begin{bmatrix} \mathbf{x} & y \\ y & z \end{bmatrix} \quad (33)$$

We can represent the space of matrices $\mathcal{S}_+^2$ of the form $S \in \mathcal{S}_+^2$ as a three dimensional space with non-negative $\mathbf{x}$, y and z coordinates and a non-negative determinant. This space



corresponds to a cone as shown in the Figure above.

Positive semidefinite cone: Notes

1. $S_+^n = \{A \in S^n | A \succeq 0\} = \{A \in S^n | \mathbf{y}^T A\mathbf{y} \succeq 0 \forall \|\mathbf{y}\| = 1\}$
2. So, $S_+^n = \cap_{\|y\|=1} \{A \in S| < \mathbf{y}^T\mathbf{y}, A > \succeq 0\}$
3. $\mathbf{y}^T A\mathbf{y} = \sum_i \sum_j y_i a_{ij} y_j = \sum_i \sum_j (y_i y_j) a_{ij} = < \mathbf{y}\mathbf{y}^T, A > = tr((\mathbf{y}\mathbf{y}^T)^T A) = tr(\mathbf{y}\mathbf{y}^T A)$
   - H/W:

$$\mathbf{y} = \begin{bmatrix} Cos(\theta) \\ Sin(\theta) \end{bmatrix} \tag{34}$$

$$\mathbf{y}\mathbf{y}^T = \begin{bmatrix} Cos^2(\theta) & Cos(\theta)Sin(\theta) \\ Cos(\theta)Sin(\theta) & Sin^2(\theta) \end{bmatrix} \tag{35}$$

   - Plot a finite # of halfspaces parameterized by (θ).
4. So S_+^n = intersection of infinite # of half spaces belonging to $\Re^{n(n+1)/2}$ [Dual Representation]

Positive semidefinite cone: Notes

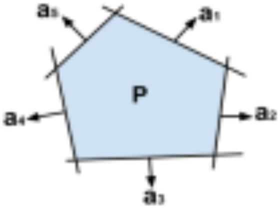
1. S_+^n = intersection of infinite # of half spaces belonging to $R^{n(n+1)/2}$ [Dual Representation]
    1. Cone boundary consists of all singular p.s.d. matrices having at-least one 0 eigenvalue.
    2. Origin = O = matrix with all 0 eigenvalues.
    3. Interior consists of all full rank matrices A (rank A = m) i.e. $A \succ 0$.

Polyhedra

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} = \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality

Specifying intersection of half spaces

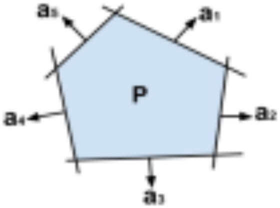
Like specifying some hyperplanes



- Intersection of finite number of half-spaces and hyperplanes.
- Question:Can you define convex polyhedra (or polytope) in terms of convex hull? Leads to definition of simplex.

Polyhedra

- Solution set of finitely many inequalities or equalities: $A\mathbf{x} \preceq \mathbf{b}$, $C\mathbf{x} \equiv \mathbf{d}$
  - $A \in \Re^{m \times n}$
  - $C \in \Re^{p \times n}$
  - $\preceq$ is component wise inequality



- Intersection of finite number of half-spaces and hyperplanes.
- Question:Can you define convex polyhedra (or polytope) in terms of convex hull? Leads to definition of simplex.
- Ans: If $\exists\ S \subset$ P s.t. $|S|$ is finite and $P = conv(S)$, then P is a polytope.
- Simplex: An n - dimensional simplex is $conv(S)$ where S is affinely independent set of $n+1$ points.

Convex combinations Generalized

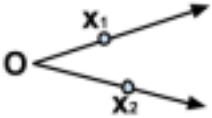
- **Convex combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k = conv(\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\})$$

$$\text{with} \quad \theta_1 + \theta_2 + ... + \theta_k = 1, \theta_i \geq 0.$$

- Equivalent Definition of Convex Set: C is convex iff it is closed under generalized convex combinations.
- **Convex hull or** $conv(S)$ is the set of all convex combinations of point in the set S.
- $conv(S)$ = The smallest convex set that contains S. S may not be convex but $conv(S)$ is.
  - Prove by contradiction that if a point lies in another smallest convex set , and not in $conv(S)$, then it must be in $conv(S)$.
  -  
- The idea of convex combination can be generalized to include infinite sums, integrals, and, in most general form, probability distributions.

Conic combinations generalized



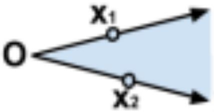
- **cone** A set C is a cone if $\forall \mathbf{x} \in C$, $\theta\mathbf{x} \in C$ for $\theta \geq 0$.
- **conic (nonnegative) combination** of points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ is any point $\mathbf{x}$ of the form

$$\mathbf{x} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + ... + \theta_k\mathbf{x}_k$$

$$\text{with} \quad \theta_i \geq 0.$$

example : diagonal vector of a parallelogram is a conic combination of two vectors(points) $\mathbf{x}_1$ and $\mathbf{x}_2$ forming the parallelogram.

Conic hull and Affine hull

- **Conic hull or conic(S):** The set that contains all conic combinations of points in set S.
- conic(S) = Smallest conic set that contains S.
- 

- Similarly, **Affine hull or aff(S):** The set that contains all affine combinations of points in set S.
- aff(S) = Smallest affine set that contains S.
-